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A NONLINEAR SYMMETRIC EIGENVALUE PROBLEM

By

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ABSTRACT

Let $K(l)$ be a $n \times n$ symmetric matrix whose elements are functions of a non-negative parameter l. The aim of this paper is to prove the convergence of an iterative method to determine the smallest value l_b and the associate vector $\mathbf{v}_b$ such that $\mathbf{K}[l_b]\ \mathbf{v}_b = 0$.

INTRODUCTION

In structural mechanics, the structures that are subject to increasing loads in a quasi static way present, with the growth of the load parameter l, a tipically nonlinear behaviour. This behaviour is characterized from the fact that it is not possible to increase the load without bounds. In fact, the answer of the structure does n't balance the external load when l is greater than a limit value, namely the collapse value.

In order to determine this collapse value, it needs the following nonlinear symmetric eigenvalue problem:

If $\mathbf{K}[l]$ is a $n \times n$ symmetric matrix whose elements are continuous functions of a nonnegative parameter l, find the smallest value l_b and the associate vector $\mathbf{v}_b$ such that

$$\mathbf{K}[l_b]\ \mathbf{v}_b = 0. \tag{1}$$

This problem also arises in structural mechanics to determnine the natural frequencies of vibration of any linearly elastic structure. In both the cases (Eulerian buckling load and frequencies of vibration) the elemets k_{ij} of $\mathbf{K}[l]$ (which represents the tangent stiffness matrix of the structure) are trascendental functions of l.

A general method for solving the problem (1) in referring to a vibration problem was already proposed in 1971 by Wittrick–Williams [1].

The numerical behaviour of the existence methods until 1973 is discussed in the survey of Ruhe [2] and successively an iterative method was developed in 1985 by Numaier [3].

Here, we are interested in an iterative method proposed by Casciaro-Aristodemo [4] in 1977 and recently, in 1989, modified with the introduction of a suitable acceleration coefficient by Casciaro-Salerno-Lanzo [5]. Precisely, in order to solve numerically the problem (1), in [5] the following iterative method is proposed:

Initialization

$$v_0 = \mathbf{K}[0]^{-1}b, \quad l_0 = (\mathbf{v}_0^t\mathbf{K}[0]\mathbf{v}_0)(\mathbf{v}_0^t\mathbf{K}'[0,\mathbf{v}_0)^{-1}$$

where $\mathbf{b} = (1, \ldots, 1,)$ and $\mathbf{K}'[0]$ is the derivative of $\mathbf{K}[l]$ in 0.

Iteration scheme

$$\mathbf{v}_{j+1} = \mathbf{v}_j - \omega_j\mathbf{K}[0]^{-1}\mathbf{K}[l]\mathbf{v}_j$$

$$l_{j+1} = \|\mathbf{v}_{j+1}\|^{-1}$$

where a suitable choice for the acceleration coefficient ω_j is

$$\omega_j = \|\mathbf{v}_j\|(4\|\mathbf{K}[0]^{-1}\mathbf{K}[l_j]\mathbf{v}_j\| + \|\mathbf{v}_j\|)^{-1}$$

Convergence condition

$$l_b = l_{j-1},\ \mathbf{v}_b = \mathbf{v}_{j+1} \text{ if } \|\mathbf{v}_{j+1} - \mathbf{v}_j\| < t\,\|\mathbf{v}_j\|$$

where t is a fixed tollerance (generally 0.000001).

In [5] the authors conjecture the convergence at the solution under the following assumptions on the eigenvalues $k_i(l)$, $i = 1, \ldots, n$, of the matrix $\mathbf{K}[l]$:

$$\mathbf{K}_1(l)\begin{cases} \epsilon\ (0,2) & \text{if } 1 < 1 \\ = 0 & \text{if } 1 = 1 \\ < 0 & \text{if } 1 > 1 \end{cases}$$

$$\mathbf{K}_{i \neq 1}(l)\begin{cases} \epsilon(0,2) & \text{if } 1 < 1 + \varepsilon,\ \varepsilon > 0 \\ < 0 & \text{if } 1 > l' \text{ and } \mathbf{K}_i(1') < 0 \\ > 2 & \text{if } 1 > l'' \text{ and } \mathbf{K}_i(1'') > 2. \end{cases}$$

This conjecture is false in general, as one easily can see in R^2 also. Neverthless, in [5] the numerical performance of the method is likely and advantageousely applied to a large number of problems in structural mechanic.

The aim of this paper is to give a complete proof, using elementary techniques, of the convergence of the algorithm under suitable hypotheses on the eigenvalues $\mathbf{K}_i(1)$.

RESULTS

With a suitable change of the variables, without loos of generality, in the following we can assume $\mathbf{K}[0]=I$ and $l_b=1$. Under these conditions, we consider the nonlinear operator T: $R^n \{0\} \rightarrow R^n$ defined by

$$T\mathbf{v} := (I - \omega \mathbf{K}[\,\|\mathbf{v}\|^{-1}])\mathbf{v}.$$

For $l>0$, let

$$\mathbf{K}[l] = \sum k_i(l)\mathbf{w}_i(l)\,\mathbf{w}_i^t(l)\ ,\ \mathbf{w}_i^t(l)\,\mathbf{w}_j(l) = \delta_{ij}$$

be the spectral representation of the symmetric matrix $\mathbf{K}[l]$.
Putting

$$\mathbf{v} = \sum (\mathbf{v})_i \omega_i\,(1/\,\|\mathbf{v}\|\,)$$

it follows that

$$T\mathbf{v} = \sum (i - \omega k_i(1/\,\|v\|\,)(\mathbf{v})_i\omega_i(1/\,\|\mathbf{v}\|\,).$$

LEMMA. With the above notations, suppose that :

(a) $l \rightarrow \mathbf{K}[l]$ is a continuous map.

(b) If l_0 is such that 0 is an eigenvalue of $\mathbf{K}[l_0]$, then the relative eigenspace is monodimensional.

(c) $$k_1(l) \begin{cases} \epsilon(0,1) & \text{if } 0<l<1 \\ =0 & \text{if } l=1 \\ <0 & \text{if } l=1. \end{cases}$$

(d) $\limsup\limits_{l\rightarrow\infty} k_1(l) = -h,\ h>0.$

(e) $k_i(l) \in (0,2)$ if $l<1+\varepsilon$, $\varepsilon>0$, for $i \neq 1$.

(f) The maps $l \rightarrow k_i(l)$ are derivables, $i=1,\ldots,n$, and $\sup|k_i(l)|<\infty$.

(g) $w \leqslant \min\,(1,\ 1/\sup|k_i'(l)|)$, $i=1,\ldots,n$.

(h) $\mathbf{w}_1(l)=\mathbf{w}_1(1)$ for all l.

Then, for each $\mathbf{v} \in R^n\{0\}$ such that $(\mathbf{v})_1 \neq 0$, we have $T^n\mathbf{v} \xrightarrow[n\to\infty]{} \mathbf{w}_1(1)$ or $T^n\mathbf{v} \xrightarrow[n\to\infty]{} -\mathbf{w}_1(1)$.

PROOF. We proceed by four steps.

STEP 1. $(T\mathbf{v}=\mathbf{v})$ iff (there exist $i \in \{1,\ldots,n\}$, $l_i<0$ such that $k_i(l_i) = 0$ and $\mathbf{v} = \pm l_i^{-1}\mathbf{w}_i\, l_i)$.

Indeed, if $\mathbf{v}=T\mathbf{v}$, putting $\mathbf{v}=\Sigma(\mathbf{v})_i\mathbf{w}_i(1/\|\mathbf{v}\|)$, we have $(1-\omega k_i(1/\|\mathbf{v}\|))(\mathbf{v})_i$, for $i=1,\ldots,n$. If $(\mathbf{v})_i \neq 0$, it follows $k_i(1/\|\mathbf{v}\|)=0$. From hypothesis (b), there exists an unique index i such that $k_i(1/\|\mathbf{v}\|)=0$. Hence the thesis follows putting $l_i=1/\|\mathbf{v}\|$. The other implication is trivial.

STEP 2. Let $\|\mathbf{v}\|>1$. If $\|T^n\mathbf{v}\| \geqslant 1$ for any n, then $T^n\mathbf{v} \xrightarrow[n\to\infty]{} \mathbf{w}_1(1)$ or $T^n\mathbf{v} \xrightarrow[n\to\infty]{} -\mathbf{w}_1(1)$.

By contradiction, we suppose that $\|T^n\mathbf{v}\| \not\to 1$. Then, there exist an $s>0$ and a subsequence $(T^{n_h}\mathbf{v})$ such that

$$\|T^{n_h}\mathbf{v}\| \geqslant 1+s \tag{4}$$

On the other hand, $(\|T^n\mathbf{v}\|)$ is a decreasing real sequence and so $\|T^n\mathbf{v}\| \xrightarrow[n\to\infty]{} a := \inf\|T^n\mathbf{v}\|$, $a>1$. Moreover, the sequence $(T^{n_h}\mathbf{v})$ is a bounded sequence and so admits a subsequence $(T^{n_{h_j}}\mathbf{v})$ convergent (for $j\to\infty$) say to $\mathbf{v}_0$. Then $\|T^{n_{h_j}}\mathbf{v}\| \xrightarrow[j\to\infty]{} \|\mathbf{v}_0\| = a$ and this is not possible by hypotheses (c) and (e). Summarizing, $\|T^n\mathbf{v}\| \to 1$.

We show now that $T^n\mathbf{v} \not\to \mathbf{w}_1(1)$ or $T^n\mathbf{v} \to -\mathbf{w}_1(1)$.

Suppose that $T^n\mathbf{v} \not\to \mathbf{w}_1(1)$. Then there exist $s_1>0$ and a subsequence $(T^{n_h}\mathbf{v})$ such that $\|T^{n_h}\mathbf{v}-\mathbf{w}_1(1)\| \geqslant s_1$. As above, one deduce the existence of a subsequence $(T^{n_{h_j}}\mathbf{v})$ of $(T^{n_h}\mathbf{v})$ necessarily convergent to $\mathbf{w}_1(1)$. Form the hypothesis (h),

$$(T^n \mathbf{v})_1 = (\mathbf{v})_1 \prod_{r=0}^{n-1} (1 - \omega k_1(1/ \| T^r \mathbf{v} \|))$$

so that, from hypothesis (c), the real monotone seqence $((T^n \mathbf{v})_1)$ has constant sign. On the other hand, $(T^{n_{h_j}} \mathbf{v})_1 \to (-\mathbf{w}_1(1))_1 = -1$. Thus, from the monotonicity of $((T^n \mathbf{v})_1)$, it follows $(T^n \mathbf{v})_1 \to -1$. From this, by standard argumentations, it follows $T^n \mathbf{v} \to \mathbf{w}_1(1)$.

STEP 3. If $\| \mathbf{v} \| \leqslant 1$ then $\| T\mathbf{v} \| \leqslant 1$.

Indeed, obviously, $\| T\mathbf{v} \| \leqslant \max_i | 1 - \omega k_i (1/ \| \mathbf{v} \|) \| \mathbf{v} \|$. So that it is enough to prove that $\max_i |1 - \omega k_i(1/ \| \mathbf{v} \|)| \leqslant 1$. At this purpose, we show that the n functions $f_i(l) = |1 - \omega k_i(l)| - l$, $i = 1, \dots, n$, satisfy $f_i(l) \leqslant 0$ for $l \geqslant 1$.

Now, of course, $f_i(1) \leqslant 0$, hence it is fufficient to show that $f'_i(1) \leqslant 0$. This follows from hypothesis (g).

STEP 4 If $(\mathbf{v})_1 \neq 0$, then $T^n \mathbf{v} \to \mathbf{w}_1(1)$ or $T^n \mathbf{v} \to -\mathbf{w}_1(1)$.

Indeed, from step 2 we may take $\mathbf{v}$ with $\| \mathbf{v} \| < 1$. First of all, we show that $\| T^n \mathbf{v} \| \to 1$. By contradiction, suppose that $\| T^n \mathbf{v} \| \not\to 1$. Then there exist $s_2 > 0$ and a subsequence $(T^{n_h} \mathbf{v})$ such that

$$\| T^{n_h} \mathbf{v} \|^{-1} \geqslant 1 + s_2 \tag{5}$$

and, without loss of generality, from pypothesis (d), we may assume

$$-k_1(1 + s_2) \leqslant -k_1(l) \quad \text{for } l \geqslant 1 + s_2.$$

Then

$$|(T^{n_1+1} \mathbf{v})_1| = |(\mathbf{v})_1| \left(\prod_{r=0}^{n_1-1} (1 - \omega k_1(1/ \| T^r \mathbf{v} \|)) \right) (1 - k \omega_1(1/ \| T^{n_1} \mathbf{v} \|)) \geqslant$$

$$\geqslant |(\mathbf{v})_1| (1 - \omega k_1(1/ \| T^{n_1} \mathbf{v} \|)) \geqslant \text{(from (5) and (6))} \geqslant$$

$$\geqslant |(\mathbf{v})_1| (1 - \omega k_1(1 + s_2)).$$

Analogously, by induction, one can see that

$$|(T^{n_h+1}\mathbf{v})_1| \geqslant |(\mathbf{v})_1|(1-\omega k_1(1+s_2))^h.$$

Now, the second member tends to ∞ as $h \to \infty$ contradicting step 3 because, of course, $\| T^{n_h+1}\mathbf{v} \| \geqslant |(T^{n_h+1}\mathbf{v})_1|$.

Thus, we have proved that $\| T^n\mathbf{v} \| \to 1$. At this point, one can proceed as in step 2.

Before giving the main result, we introduce the following notations:

$$AV(\| \hat{\mathbf{v}} \|, \| T\hat{\mathbf{v}} \|, \mathbf{v}) := \mathbf{v}^t\mathbf{w}_1(1/\| T\hat{\mathbf{v}} \|) - \mathbf{v}^t\mathbf{w}_1(1/\| \hat{\mathbf{v}} \|)$$

and

$$AV(\| \hat{\mathbf{v}} \|, \| T\hat{\mathbf{v}} \|) := max\{ |AV(\| \hat{\mathbf{v}} \|, \| T\hat{\mathbf{v}} \|, \mathbf{v}) : \|\mathbf{v}\| = \|T\hat{\mathbf{v}}\| \}.$$

THEOREM. Suppose that the hypotheses (a)-(g) of the previous Lemma are verified. Moreover suppose that there exists $D>0$ such that:

(i) $\mathbf{w}_1(l) = \mathbf{w}_1(1)$ for $l \in [1, 1+D]$

(ii) $k_1(l) \leqslant k_1(1+D) = : -d$ for $l \geqslant 1+D$

(iii) $AV(\| \mathbf{v} \|, \| T\mathbf{v} \|) \leqslant d/4$ for $\| \mathbf{v} \| < 1$

(iv) $k_1(l)$ is a decreasing function in $[1, 1+D]$.

Then. if $\mathbf{v}$ is a vector such that $\| \mathbf{v} \| < 1$ and $|(\mathbf{v})_1| > 1/2$, we have $T^n\mathbf{v} \to \mathbf{w}_1(1)$ or $T^n\mathbf{v} \to -\mathbf{w}_1(1)$.

PROOF. We have

$$|(T\mathbf{v})_1| = |(1-\omega k_1(1/\| \mathbf{v} \|))(\mathbf{v})_1 + AV(\| \mathbf{v} \|, \| T\mathbf{v} \|, T\mathbf{v}) \geqslant$$
$$\geqslant |(1-\omega k_1(1/\| \mathbf{v} \|))|(\mathbf{v})_1| - AV(\| \mathbf{v} \|, \| T\mathbf{v} \|) \qquad (7)$$

We affirm that there exists an integer n_0 such that

$$1/\| T^{n_0}\mathbf{v} \| < 1+D.$$

Indeed, by contradiction, putting $|(\mathbf{v})_1| = (1/2)(1+a)$, $a>0$, the (7), (ii) and (iii) yield

$$|(T\mathbf{v})_1| \geqslant (1/2)((1+d/2)+a(1+d))$$

and, for induction,

$$|(T^n\mathbf{v})_1| > (1/2)[(1+d/2)^n = a(1+d)^n]$$

contradicting the fact that $\| T^n\mathbf{v} \| \leqslant 1$ (see step 3 of Lemma).

Thus, let n_0 be the first positive integer for which

$$1/\| T^{n_0}\mathbf{v} \| < 1 + D.$$

If $1/\| T^n\mathbf{v} \| \leqslant 1 + D$ for all $n \geqslant n_0$, we are done thanks to Lemma since in $[1, 1+D]$,

$$\mathbf{w}_1(1) = \mathbf{w}_1(l).$$

Thus let n_1 be the first positive intger greater than n_0 such that

$$1/\| T^{n_1}\mathbf{v} \| \geqslant 1 + D .$$

Then obviously

$$|T^n\mathbf{v}_1| \geqslant (1/2)[(1+d/2)^{n_0} + a(1+d)^{n_0}] \text{ if } n_0 \leqslant n \leqslant n_1 .$$

Moreover one can see that

$$|(T^{n_1}\mathbf{v})_1| \geqslant (1/2)[(1+d/2)^{n_0-1} + a(1+d)^{n_1}]$$

and

$$|(T^{n_1+1}\mathbf{v})_1| \geqslant (1/2)[(1+d/2)^{n_0} + a(1+d)^{n_0+1}].$$

Now, if $1/\| T^n\mathbf{v} \|$ is not definitely less or equal to $1+D$, we can construct the seqence of positive integers $n_1, n_2, \ldots$ inductively as above, which satisfies

$$|(T^{n_{2h-1}+1}\mathbf{v})_1| \geqslant (1/2)[(1+d/2)^{n_0} + a(1+d)^{n_0+h}]$$

but this is not possible because the second member tends to ∞ as $h \to \infty$ contradicting $\| T^n\mathbf{v} \| \leqslant 1$ for all n.

Thus, it results that $1/\| T^n\mathbf{v} \| \leqslant 1 + D$.

At this point from hypothesis (i), the proof proceed as in Lemma.

REMARK 1. If the matrix $\mathbf{K}[l]$ is linear in l, *i. e.* $\mathbf{K}[l] = I - lH$, being H a nonnegative definite matrix, and $\omega = 1$, then the eigenvalues $k_i(l)$ have the form $k_i(l) = 1 - lh_i$, $h_i > 0$ and the basis $\{\mathbf{w}_1(l), \ldots, \mathbf{w}_n(l)\}$ is constant in l. This, our theorem includes the classic linear case.

REMARK 2. It is worth observing that

$$AV(\|\mathbf{v}\|, \|T\mathbf{v}\|) \leqslant \|T\mathbf{v}\| \, max \, \{ |a(\hat{\mathbf{v}}, \mathbf{w}_1(1/\|T\mathbf{v}\|)) - a(\hat{\mathbf{v}}, \mathbf{w}_1(1/\|\mathbf{v}\|))| : \|\hat{\mathbf{v}}\| = \|T\mathbf{v}\| \}$$

where $a(\mathbf{v}_1, \mathbf{v}_2)$ denotes the angle between the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. So the hypothesis (iii) of our theorem is certainly satisfied when $\|T\mathbf{v}\|$ is small enough.

While the hypotheses (i), (ii) and (iv) arise naturally in the numerical preformance for solving many problems of structural mechanic, the condition (iii) is actually restrictive. It is easy to construct examples for which the thesis of the theorem is false if (iii) is not verified. Neverthless it is not a necessary condition, of course, and can be certainly more conveniently improved.

Finally, we underline that in many cases the engineers know the required solution with a "good" approximation, so that the initial condition $|(\mathbf{v})_1| > 1/2$ of our theorem, is not a bad condition.

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GENERALIZATION OF HAIMABATI RESULT

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ABSTRACT

Here we try to generalize the result of HAIMABATI CHATTERJI (1980) which extends some results of Banach (1922), Kannan (1968), Reich (1971), Chatterji (1972), Hardy and Rogers (1973) and Ciric (1971).

THEOREM . Let $\{f_n\}$ $(n = 0, 1, 2, \ldots)$ be sequence of mappings of a complete metric space X into itself such that for some h with $0 < h < 1$ and for every $x, y \in X$,

$$d(f_0^p(x), f_n^q(y)) \leqslant h \max\Big\{ d(x,y), d(x, f_0^p(x)), d(y, f_n^q(y)), [d(x, f_n^q(y)) + d(y, f_0^p(x))]/2\Big\} \quad \ldots (1)$$

for each $n = 1, 2, \ldots$ and $p, q \geqslant 1$ hold. Then there exists a unique common fixed point of f_n $(n = 0, 1, 2, \ldots,)$ in X.

Proof: Consider $x_0 \in X$. Then we construct a sequence $\{x_n\}$ as follows:

x_0, $x_1 = f_0^p(x_0)$, $x_2 = f_1^q(x_1)$, $x_3 = f_0^p(x_2)$, $x_4 = f_2^q(x_3)$, $\ldots$,

$x_{2n-1} = f_0^p(x_{2n-2})$, $x_{2n} = f_n^q(x_{2n-1}), \ldots,$

Now,
$$\begin{aligned} d(x_1, x_2) &= d(f_0^p(x_0), f_1^q(x_1)) \\ &\leqslant h \max\Big\{ d(x_0, x_1), d(x_0, f_0^p(x_0)), d(x_1, f_1^q(x_1)), \\ &\qquad [d(x_0, f_1^q(x_1)) + d(x_1, f_0^p(x_0))]/2 \Big\} \\ &\leqslant h \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \\ &\qquad [d(x_0, x_2) + d(x_1, x_1)]/2\} \end{aligned}$$

$$= h\, max\ \{\, d(x_0,x_1)\,,\,[d(x_1,x_2)\,,\, d(x_0,x_2)/2\}$$

$$\leqslant h\, max\ \{d(x_0,x_1)\,,\, d(x_1,x_2)\,,\,[\, d(x_0,x_1) + d(x_1,x_2)]/2\} \qquad \dots (2)$$

If $d(x_1,x_2)$ is maximum, then from (2),

$$d(x_1, x_2) \leqslant h\, d(x_1, x_2) < d(x_1, x_2)\,, \text{ a contradiction.}$$

Thus $d(x_0,x_1) \geqslant d(x_1,x_2)$ ... (3)

Again if possible, let

$$d(x_0,x_1) < [\, d(x_0,x_1) + d(x_1,x_2)]/2$$

i. e., $d(x_0,x_1) < d(x_1,x_2)$ which is not possible by (3). Hence from (2) we get

$$d(x_1,x_2) \leqslant h\, d(x_0,x_1)$$

Again, $d(x_2,x_3) = d(\overset{p}{f}_0(x_2)\,,\, \overset{q}{f}_1(x_1))$

$$\leqslant h\, max\{\, d(x_2,x_1)\,,\, d(x_2,x_3)\,,\, d(x_1,x_2)\,,\ [\, d(x_2,x_2) + d(x_1,x_3)]/2\}$$

$$\leqslant h\, max\{\, d(x_1,x_2),\, d(x_2,x_3),\, d(x_1, x_3)/2\} \qquad \dots (4)$$

By the similar argument, we get from (4)

$$d(x_2, x_3) \leqslant h\, d(x_1, x_2) \leqslant h^2\, d(x_0, x_1)\ .$$

Therefore $\quad d(x_n, x_{n+1}) \leqslant h^n\, d(x_0, x_1)\ .$

Now, $d(x_n, x_{n+m}) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+m-1}, x_{n+m})$

$$\leqslant (h^n + h^{n+1} + \dots + h^{n+m-1}\, d(x_0, x_1)$$

$$\leqslant \frac{h^n}{1-h}\, d(x_0, x_1) \to 0 \text{ as } n \to \infty\ .$$

Thus $\{x_n\}$ is a Cauchy sequence and since X is complete

$$\lim_{n \to \infty} x_n = \xi \in X.$$

Thus, $d(\xi, \overset{p}{f}_0(\xi)) \leqslant d(\xi, x_{2n}) + d(\overset{p}{f}_0(\xi)\,,\, \overset{q}{f}_n(x_{2n-1}))$

$$\leqslant d(\xi\,,\, x_{2n}) + h\, max\ \Big\{\, d(\xi, x_{2n-1}),\ d(\xi\,,\, \overset{p}{f}_0(\xi)),$$

$$d(x_{2n-1}, \overset{p}{f}_n(x_{2n-1})),$$

$$[\, d(\xi, f_n^q(x_{2n-1})) + d(x_{2n-1}, f_0^p(\xi))]/2 \Big\}$$

$$\leqslant d(\xi\,, x_{2n}) + h\; max\; \Big\{ d(\xi\,, x_{2n-1}), d(\xi\,, f_0^p(\xi)),$$

$$d(x_{2n-1}, x_{2n}), [d(\xi\,, x_{2n}) + d(x_{2n-1}, f_0^p(\xi))]/2 \Big\}.$$

When $n \to \infty$ we get,

$$d(\xi\,, f_0^p(\xi)) \leqslant h\; max \Big\{ 0, 0, d(\xi\,, f_0^p(\xi)), d(\xi\,, f_0^p(\xi))/2 \Big\}.$$

Thus from this we conclude that $\xi = f_0^p(\xi)$,

i. e., ξ is the fixed point of f_0^p.

Now we have to show that ξ is also fixed point of f_n^q $(n = 1, 2, \ldots, n)$.

Since, $d(\xi\,, f_n^q(\xi)) = d\, f_0^p(\xi)\,, f_n^q(\xi))$

$$\leqslant h\; max\; \{d(\xi\,, \xi)\,, d(\xi. \xi)\,,$$

$$d(\xi, f_n^q(\xi)), [d(\xi, f_n^q(\xi)) + d(\xi, \xi)]/2\Big\}.$$

We get from this that $\xi = f_n^q(\xi)$.

i. e., ξ is also the fixed point of f_n^q.

Further we have to show that ξ is unique fixed point of f_0^p and $(f_n^q\, (n = 1, 2, \ldots)$.

If not, suppose ξ and η be their two fixed points such that

$$\xi \neq \eta \text{ and } f_0^p(\eta) = f_n^q(\eta) = \eta.$$

Then, $d(\xi, \eta) = d(f_0^p(\xi). f_n^q(\eta))$

$$\leqslant h\; max \Big\{ d(\xi, \eta), d(\xi. f_0^p(\xi)), d(\eta\,, f_n^q(\eta))\,,$$

$$[\, d(\xi. f_n^q(\eta)) + d(\eta. f_0^p(\zeta))]/2 \Big\}$$

$$\leqslant h\ max\ \{d\ \xi, \eta)\ ,\ d(\xi, \xi),\ d(\eta, \eta),$$

$$[\ d(\xi, \eta) + d(\eta, \xi)]/2\}$$

which implies that $\xi = \eta$.

Thus ξ is the unique fixed point of f_0^p and f_n^q $(n = 1, 2, 3, \ldots)$.

Again we have to show that ξ is the unique common fixed point of f_n $(n = 0, 1, 2, \ldots)$.

For , $f_0^p(f_0(\xi)) = f_0(f_0^p(\xi))$ give ,

$$f_0^p(f_0(\xi)) = f_0(\xi) \ .$$

i. e., $f_0(\xi) = \xi$, by the uniqueness of ξ as the fixed point of f_0^p . Similarly $f_n^q f_n(\xi)) = f_n(f_n^q(\xi))$ give,

$$f_n^q(f_n(\xi)) = f_n(\xi) \ .$$

i. e., $f_n(\xi) = \xi$, by the uniqueness of ξ as the fixed of f_n^q

Thus ξ is the fixed point of f_n $(n = 0, 1, 2, \ldots)$.

Finally, we show that ξ is the only fixed point common to f_n $(n = 0, 1. 2, \ldots)$.

For, if ζ were a point such that $\xi \neq \zeta$ and $f_n(\zeta) = \zeta$. Then,

$$d(\xi, \zeta) = d(f_0(\xi),\ f_n(\zeta))$$

$$= d(f_0^p(\xi)\ ,\ f_n^q(\zeta))$$

$$\leqslant h\ max\Big\{\ d(\xi, \zeta),\ d(\xi, f_0^p(\xi)),$$

$$d(\zeta, f_n^q(\zeta)),\ [\ d(\xi, f_0^q(\zeta)) + d(\zeta, f_0^p(\xi))]/2\Big\}$$

$$\leqslant h\ max\ \{d(\xi, \zeta),\ d(\xi, \xi),\ d(\zeta, \zeta),\ [\ d(\xi, \zeta) + d(\zeta, \xi)]/2\}$$

Showing that $\xi = \zeta$.

Remarks : If we put $p = q = 1$ in our theorem (without considering f_0 continuous), then we get the following theorem of HAIMABATI CHATTERJI

Let $\{ f_n \}$ $(n = 1, 2, \ldots)$ be a sequence of mappings of a complete metric space X into itself. If the following conditions:

(i) f_0 is continuous .

(ii) for some q weth $0 < q < 1$ and for every $x, y \in X$,

$$d(f_0(x), f_n(y)) \leqslant q \max\{d(x,y), d(x, f_0(x)), d(y, f_n(y)), [\, d(x, f_n(y)) + d(y, f_0(x))/2]\}$$

for each $n = 1, 2, \ldots$ hold. Then there exists a unique common fixed point of $f_n (n = 0, 1, 2, \ldots)$ in X .

THEOREM 2. Let $\{ f_m \}$, $\{g_n\}$ $(m, n = 1, 2, 3, \ldots)$ be two sequences of mappings on a complete metric space X into intself. If for some h with $0 < h < 1$ and for every $x, y \in X$,

$$d(f^p_m (x) , g^q_n (y)) \leqslant h \max \left\{ d(x, y), d(x, f^p_m (x)), d(y , g^q_n (y)), [d(x, g^q_n (y)) + d(y, f^p_m (x))]/2 \right\} \quad \ldots(1)$$

for each $m,n = 1,2,3, \ldots$ with $p,q \geqslant 1$ hold. Then f_m, g_n $(m,n = 1,2,3,\ldots)$ each have a unique common fixed point .

Proof : Consider any $x_0 \in X$ and construct a sequence $\{x_n\}$ as follows:

$$x_0, x_1 = f^p_n (x_0), x_2 = g^q_1 (x_1), x_3 = f^p_2 (x_2), x_4 = g^q_2 (x_3), \ldots$$

$$x_{2n-1} = f^p_n (x_{n-2}), x_{2n} = g^q_n (x_{2n-1}) .$$

Now,

$$\begin{aligned} d(x_1, x_2) &= d(f^p_1 (x_0), g^q_1 (x_1)) \\ &\leqslant h \max\Big\{ d(x_0, x_1) , d(x_0, f^p_1 (x_0)), d(x_1, g^q_1 (x_1)), \\ &\qquad [\, d(x_0, g^q_1 (x_1)) + d(x_1, f^p_1 (x_0))/2 \,]\Big\} \\ &= h \max\{ d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), \\ &\qquad [\, d(x_0\, x_2) + d(x_1,x_1)/2]\} \\ &\leqslant h \max \{d(x_0,x_1), d(x_1\, x_2), [d(x_0,x_1) + d(x_1,x_2)/2]\} \quad \ldots (2) \end{aligned}$$

If $d(x_1,x_2)$ will be maximum, then from (2),

$$d(x_1,x_2) \leqslant h\, d(x_1,x_2) < d(x_1,x_2), \text{ a contradiction.}$$

Thus, $d(x_0,x_1) \geqslant d(x_1,x_2)$ ...(3)

Again if possible, let $d(x_0,x_1) < [\, d(x_0,x_1) + d(x_1,x_2)]/2$ *i. e.*, $d(x_0,x_1) < d(x_1,x_2)$ which is not possible by (3). Hence from (2) we get $d(x_1,x_2) \leqslant h\, d(x_0, x_1)$.

Again,

$$\begin{aligned} d(x_2,x_3) &= d(g_1^q(x_1), f_2^p(x_2)) \\ &= d(f_2^p(x_2), g_1^q(x_1)) \\ &\leqslant h\, max\Big\{d(x_2, x_1),\ d(x_2, f_2^p(x_2)),\ d(x_1, g_1^q(x_1)), \\ &\qquad [\, d(x_2, g_1^q(x_1)) + d(x_1, f_2^p(x_2))]/2\Big\} \\ &= h\, max\{d(x_1,x_2), d(x_2,x_3), d(x_1,x_2), \\ &\qquad [\, d(x_2,x_2), + d(x_1,x_3)]/2\} \\ &\leqslant h\, max\{d(x_1,x_2), d(x_2,x_3), [d(x_1,x_2) + d(x_2,x_3)]/2\} \quad \dots(4) \end{aligned}$$

By the similar process, we get from (4),

$d(x_2,x_3) \leqslant h\, d(x_1,x_2) \leqslant h^2\, d(x_0,x_1)$

Hence, $d(x_n, x_{n+1} \leqslant h^n\, d(x_0,x_1)$

Now,

$$\begin{aligned} d(x_n,x_{n+t}) &\leqslant (x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \dots + d(x_{n+t-1},x_{n+t}) \\ &\leqslant (h^n + h^{n+1} + \dots + h^{n+t-1})\, d(x_0,x_1) \\ &< \frac{h^n}{1-h}\, d(x_0,x_1) \to 0 \text{ as } n \to \infty . \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence and since X is complte $\lim_{n\to\infty} x_n = \xi \in X$.

Thus $d(\xi, f_m^p(\xi)) \leqslant d(\xi,x_{2n}) + d(f_m^p(\xi),\ g_n^q(x_{2n-1}))$

$\leqslant d(\xi,x_{2n}) + h\ max\Big\{ d(\xi,x_{2n-1}),\ d(\xi, f_m^p(\xi)),\ d(x_{2n-1}, g_n^q(x_{2n-1}))$

$$[d(\xi, g^{q}_{n}(x_{2n-1})) + d(x_{2n-1}, f^{p}_{m}(\xi))]/2 \Big\}$$

$$\leqslant d(\xi, x_{2n}) + h \; max \Big\{ d(\xi, x_{2n-1}), d(\xi, f^{p}_{m}(\xi)),$$

$$d(x_{2n-1}, x_{2n}), [d(\xi, x_{2n}) + d(x_{2n-1}, f^{p}_{m}(\xi))]/2\} .$$

When $n \to \infty$ we get

$$d(\xi, f^{p}_{m}(\xi)) \leqslant h \; max \Big\{0, 0, d(\xi, f^{p}_{m}(\xi)), [d(\xi, f^{p}_{m}(\xi))]/2\Big\} .$$

Thus from this we conclude that $\xi = f^{p}_{m}(\xi)$.

i. e., ξ is the fixed point of f^{p}_{m} .

Now we have to show that ξ is also the fixed point of g^{q}_{n} $(n = 1, 2, \ldots, n)$.

Since,

$$d(\xi, g^{q}_{n}(\xi)) = d(f^{p}_{m}(\xi), g^{q}_{n}(\xi))$$

$$\leqslant h \; max\Big\{d(\xi, \xi), d(\xi, \xi), d(\xi, g^{q}_{n}(\xi)),$$

$$[d(\xi, g^{q}_{n}(\xi)) + d(\xi, \xi))]/2\Big\}$$

We get from this that $\xi = g^{q}_{n}(\xi)$. *i. e.*, ξ is also fixed point of g^{q}_{n} .

Now we show that ξ is unique fixed point of f^{p}_{m} and g^{q}_{n} .

If not, suppose, ξ, η be their two fixed points such that

$\xi \neq \eta$ and $f^{p}_{m}(\eta) = g^{q}_{n}(\eta) = \eta$.

Then $d(\xi, \eta) = d(f^{p}_{m}(\xi), g^{q}_{n}(\eta))$

$$\leqslant max\Big\{d(\xi, \eta), d(\xi, f^{p}_{m}(\xi)), d(\eta, g^{q}_{n}(\eta)),$$

$$[d(\xi \cdot g^{q}_{n}(\eta)) + d(\eta, f^{p}_{m}(\xi))]/2 \Big\}$$

$$\leqslant h \; max\{d(\xi, \eta), d(\xi, \xi), (\eta, \eta), [d(\xi, \eta) + d(\eta, \xi)]/2\}$$

which implies that $\xi = \eta$. Thus ξ is the unique common fixed point of f^p_m and g^q_n.

Further we have to show that ξ is the unique common fixed point of f^p_m $(m = 1, 2, \ldots)$ and g^q_n $(n = 1, 2, 3, \ldots)$.

For, $f^p_m(f_m(\xi)) = f_m(f^p_m(\xi))$ give $f^p_m(f_m(\xi)) = f_m(\xi)$.

i. e., $f_m(\xi) = \xi$, by the uniqueness of ξ as the fixed point of f^p_m.

Similarly $g_n(\xi) = \xi$.

Finally we show that ξ is the only fixed point common to f_m $(m = 1, 2, \ldots)$ and g_n $(n = 1, 2, \ldots)$.

For, if ζ were a point such that $\xi \neq \zeta$ and $f_m(\zeta) = g_n(\zeta) = \zeta$, then

$$d(\xi, \zeta) = d(f_m(\xi), g_n(\zeta)) = d(f^p_m(\xi), g^q_n(\zeta))$$

$$\leqslant h\ max\Big\{ d(\xi. \zeta), d(\xi. f^p_m(\xi)), d(\zeta. g^q_n(\zeta)), [d(\xi, g^q_n(\zeta)) + d(\zeta, f^p_m(\xi))]/2\Big\}$$

$$\leqslant h\ max\{d(\xi\ \zeta), (\xi, \xi), d(\zeta. \zeta), [d(\xi. \zeta) + d(\zeta, \xi)]/2\}$$ shows that $\xi = \zeta$.

We can also prove the following theorem by similar method.

THEOREM 3. Let $\{U_1, \ldots, U_p\}$ and $\{V_1, \ldots, V_q\}$ be two operators such that

(i) U_i $(1 \leqslant i \leqslant p)$ afid V_μ $(1 \leqslant \mu \leqslant q)$ all map X into itself ;

(ii) $U_i U_j = U_j U_i$, where $1 \leqslant i, j \leqslant p$ and $V_\mu V_\nu = V_\nu V_\mu$ where $1 \leqslant \mu, \nu \leqslant q$;

(iii) for all $x\ y \in X$,

$$d\ U(x), V(y)) \leqslant h\ max\{d\ x,y), d(x,U(x)), d(y,V(y)), [d(x,V(y)) + d(y,U(x))]/2\}$$ with $U = U_1, \ldots, U_p$ and $V = V_1, \ldots V_p$ $0 < h < 1$. ...(1)

Then $U_i (1 \leqslant i \leqslant p)$ and V_μ $(1 \leqslant \mu \leqslant q)$ have a unique common fixed pint.

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A GENERALIZATION OF A CLASS OF POLYNOMIALS

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ABSTRACT

Motivated by several earlier works (including, for example, that by H. M. Srivastava and J. P. Singhal [12], the authors introduce and study a class of polynomials defined by equation (1.5) below.

1. INTRODUCTION

In an attempt to unify and extend the works of a number of earlier authors, Srivastava and Singhal [12] studied a class of polynomials defined by generalized Rodrigues' formula (see also [11, p.401, Problem 34]) :

$$(1.1)\ G_n^{(\alpha)}(x, r, p, k) = \frac{1}{n!}\ x^{-\alpha-kn} \exp(px^r) \left(x^{k+1}\frac{d}{dx}\right)^n \left\{x^\alpha \exp(-px^r)\right\},$$

where the parameters α, r, p and k are arbitrary, in general. Subsequently, but independently, Chandel ([1], [2]) studied a slight variation of the Srivastava-Singhal polynomials in the form :

$$(1.2)\ T_n^{(\alpha,k)}(x, r, p) = x^{-\alpha} \exp(p x^r)\ \Omega_x^n \left\{x^\alpha \exp(-px^r)\right\},$$

where, and in what follows,

$$(1.3)\ \Omega_x = x^k \frac{d}{dx}.$$

Chandel [3], on the other hand, presented a further generalization of these polynomials, defined by

$$(1.4)\ G_n(h, g, k) = e^{-h\,g(x)}\ \Omega_x^n \left\{e^{hg(x)}\right\},$$

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where h and k are independent of x. The special case $k \to 1$ of the polynomials defined by (1.4) was considered earlier by Shrivastava [10] . Moreover, the special case $k = 0$ of $G_n(h, g, k)$ may be compared with the Bell polynomials (cf. [8] and [11]) .

Motivated by above works, we introduce and study here a class of polynomials defined by

$$(1.5)\quad S_n^{(\alpha,k)}(h,g) = [1 - h\,g(x)]^{\alpha}\ \Omega_x^n \left\{(1 - h\,g(x))^{-\alpha}\right\},$$

where α, h and k are independent of x .

On replacing h by h/α, and letting $\alpha \to \infty$, (1.5) reduces to (1.4) .

2. Operational results

We can easily prove that

$$(2.1)\quad F(\Omega_x)\,\{(1 - h\,g(x))^{-\alpha}\, f(x)\}$$

$$= [1 - h\,g(x)]^{-\alpha}\ F\left(\frac{\alpha\, x^k\, h\, g'(x)}{1 - h\,g(x)} + \Omega_x\right)\{f(x)\},$$

which for $F(\Omega_x) = \Omega_x^n$, gives

$$(2.2)\quad [1 - h\,g(x)]^{\alpha}\ \Omega_x^n\ \{(1 - h\,g(x))^{-\alpha}\, f(x)\}$$

$$= \left(\frac{\alpha\, x^k\, h\, g'(x)}{1 - h\,g(x)} + \Omega_x\right)^n \{f(x)\} .$$

For $f(x) = 1$ the above result further gives

$$(2.3)\quad \odot^n\ \{1\} = S_n^{(\alpha,k)}(h,g),$$

where

$$(2.4)\quad \odot = \frac{\alpha\, x^k\, h\, g'(x)}{1 - h\,g(x)} + \Omega_x.$$

We can also prove the following generalized Leibnitz rule for the above operator :

$$(2.5)\quad \odot^n\ \{u.v\} = \sum_{m=0}^{n} \binom{n}{m}\, \Omega_x^m\ \{u\}\ \odot^{n-m}\ \{v\}.$$

Choosing $u = 1$ and making appeal to (2.3), we can obtain

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$$(2.6) \quad \odot^n = \sum_{m=0}^{n} \binom{n}{m} S_{n-m}^{(\alpha,k)} (h, g)\, \Omega_x^m .$$

Now making an appeal to (2.3) and (2.6), we derive

$$(2.7) \quad S_{n+m}^{(\alpha,k)} (h, g)$$

$$= \sum_{j=0}^{n} \binom{n}{j} S_{n-j}^{(\alpha,k)} (h, g)\, \Omega_x^j \,\{S_m^{(\alpha,k)} (h, g)\}.$$

we also derive

$$(2.8) \quad \Omega_x^n \,\{S_m^{(\alpha,k)} (h, g)\}$$

$$= \sum_{j=0}^{n} \binom{n}{j} S_{n-j}^{(\alpha,\lambda)} (h, g)\, S_{m+j}^{(\alpha,k)} (h, g) .$$

Taking $m = 0$, (2.8) further gives the following operational result :

$$(2.9) \quad \Omega_x^n = \sum_{j=0}^{n} \binom{n}{j} S_{n-j}^{(-\alpha,k)} (h, g)\, \odot^j .$$

Making an appeal to (2.3), we also ohtain

$$(2.10) \quad {}_\lambda F_\mu \left[\begin{matrix} (a_\mu); \\ (b_\mu); \end{matrix} \; t\odot \right] \{S_m^{(\alpha,k)} (h, g)\}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\lambda} (a_j)_n \, t^n}{\prod_{j=1}^{\mu} (b_j)_n \, n!} S_{n+m}^{(\alpha,k)} (h, g) , \qquad |t| < 1 ,$$

which will give several special cases .

3. Generating relation

Making an appeal to definition (1.5), we have

$$(3.1) \quad [\, 1 - h\, g(x)]^\alpha \; {}_\lambda F_\mu \left[\begin{matrix} (a_\lambda); \\ (b_\mu); \end{matrix} \; t\, \Omega_x \right] \{\{1 - h\, g(x)^{-\alpha}\}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\lambda} (a_j)_n \, t^n}{\prod_{j=1}^{\mu} (b_j)_n \, n!} S_n^{(\alpha,k)} (h, g).$$

Particularly, For $\lambda = \mu$, $a_j = b_j$, $j = 1, \ldots, \lambda$ (or μ), (3.1) reduces to

$$(3.2) \quad [1-h\,g(x)]^{\alpha}\; e^{t\Omega_x}\,\{(1-h\,g(x))^{-\alpha}\} = \sum_{n=0}^{\infty} S_n^{(\alpha,k)}(h,g)\,\frac{t^n}{n!}.$$

Now using the elementary formnla (cf.,*e.g.*,) [12], p. 76, *e.q* (1.12))

$$(3.3) \quad e^{t\Omega_x}\,\{f(x)\} = f\,[x\{1-(k-1)\,t\,x^{k-1}\}^{-1/(k-1)}],\, k \neq 1,$$

we derived the generating relation

$$(3.4) \quad (1-h\,g(x))^{\alpha}\,[1-h\,g\,(x\{1-(k-1)\,t\,x^{k-1}\}^{-1/(k-1)}]^{-\alpha}$$

$$= \sum_{n=0}^{\infty} S_n^{(a,k)}\,(h,g)\,\frac{t^n}{n!}\;.$$

Making an appeal to (2.6), we obtain

$$(3.5) \quad e^{t\odot}\,\{(f(x))\} = [1-h\,g(x)]^{\alpha}\,[1-h\,g\,(\frac{x}{[1-(k-1)\,tx^{k-1}]^{1/(k-1)}})]^{-\alpha}$$

$$f\,(\frac{x}{[\,1-(k-1)\,t\,x^{k-1}]^{1/(k-1)}})\,,$$

which for $f(x) = 1$, gives (3.4) while for $f(x) = S_m^{(\alpha'k)}(h,g)$ it gives

$$(3.6) \quad \sum_{n=0}^{\infty} S_{m+n}^{(\alpha,k)}\,(h,g)\,\frac{t^n}{(k-1)^n\,x^{(k-1)n}}\;\frac{1}{n!}$$

$$= [1-h\,g(x)]^{\alpha}\Big[1-h\,g\Big(\frac{x}{(1-t)^{1/(k-1)}}\Big)\Big]^{-\alpha}$$

$$S_m^{(\alpha,k)}\Big[\,h\;g\;(\;\frac{x}{(1-t)^{1/(k-1)}}\;)\,\Big].$$

An appeal to generating relation (3.4) gives

$$(3.7) \quad S^{(\alpha+\beta,k)}(h,g) = \sum_{m=0}^{n}\binom{n}{m} S_{n-m}^{(\alpha,k)}(h,g)\,S_m^{(\beta,k)}(h,g)\,.$$

Now making the application of the operator $\odot$, we derive the recurrence relation

$$(3.8) \quad [1-h\,g(x)]\;\;S_{n+1}^{(\alpha,k)}\,(h,g)$$

$$= \alpha\,x^k\,h\,g'(x)]\;S_n^{(\alpha,k)}\,(h,g) + (1-h\,g(x)\,)\,x^k\,DS_n^{(\alpha,k)}\,(h,g)\,,$$

which can also be obtained by generating relation (3.4).

4. An extension of (2.6) and (2.9)

Combining (2.6) and (2.9), we can derive

$$(4.1)\quad \circledS^{n}_{\alpha,h,g} = \sum_{m=0}^{n} \binom{n}{m} S^{(\alpha,k)}_{n-m}(h,g) \sum_{j=0}^{m} \binom{m}{j} S^{(-\beta,k)}_{m-j}(l,f)\, \circledS^{j}_{\beta,l,f} .$$

and

$$(4.2)\quad \circledS^{n}_{\alpha,h,g} = \sum_{m=0}^{n} \binom{n}{m} S^{(\alpha,k)}_{n-m}(h,g) \sum_{j=0}^{m} \binom{m}{j} S^{(-\beta,k)}_{m-j}(l,g)\, \circledS^{j}_{\beta,l,g} .$$

5. Explicit from

Making an appeal to the result due to Chandel [1, (2.8)]

$$\Omega^{n}_{x} \{ f(z,\lambda))\} = \sum_{s=0}^{n} \frac{(-1)^s}{s!} \frac{d^s}{dz^s} f(z) \sum_{j=0}^{s} (-1)^j \binom{s}{j} z^{s-j}\, \Omega^{n}_{x} \{ z^j \},$$

we get the following explicit form for the above polynomials :

$$(5.1)\quad S^{(\alpha,k)}_{n}(h\,g) = \sum_{s=0}^{n} \frac{(a)_s\, h^s}{s!\,[1-h\,g(x)]^s} \sum_{j=0}^{s} (-1)^j \binom{s}{j}$$

$$[g(x)]^{s-j}\, \Omega^{n}_{x} \{ (g\,x))^j \} .$$

A Wronskian for $S^{(\alpha,k)}_{n}(h,g)$

Lakshman Rao [7] gave a Wronskian in an attempt to prove Turan's inequality for Hermite polynomials . Chatterjea [4] and Singh [9] gave similar Wronskians for Hermite polynomials and generalized Hermite polynomials [6], respectively, Chandel ([2], [3]) has given Wronskians for the polynomials defined by (1.2) and (1.4) respectively Here in this section, we derive a similar Wronskian for $S^{(\alpha,k)}_{n}(h\,g)$:

Theorem. If

$$(6.1)\quad \sigma^{(\alpha,k)}_{n}(h,g) = \text{the determinant of } [\, S^{(\alpha,k)}_{n+i-j}(h,g)]$$

$$0 \leqslant i,\ j \leqslant m,$$

and

(6.2) $W\{S_n, S_{n-1}, \ldots, S_{n-m}\}$

$$= \textit{the determinant of } \Omega_n^{i}\,[\{S_{n-j}\}], \quad 0 \leqslant i,\, j = m\,.$$

Then

(6.3) $\sigma_n^{(\alpha,k)}(h,g) = W\{S_n, S_{n-1}, \ldots, S_{n-m}\}$.

Proof : Interchanging m and n in (2.7), we have

(6.4) $S_{m+n} = \sum_{j=0}^{m} \binom{m}{j} S_{m-j}\, \Omega_x^{j}\, \{S_n\}$,

which for $m = 1$, gives

(6.5) $S_{n+1} = S_1 S_n + \Omega_x \{S_n\}$.

Replacing n by $n-1$, $n-2, \ldots, n-m$ in sucession, we get the equivalent expressions for the constituents of the second row of $\sigma_n^{(\alpha,k)}(h, g)$. We shall call each of these steps a process. After the first process, we obtain

$\sigma_n^{(\alpha,k)}(h, g)$ = the determinant of the matrix (6.1) with second row having its jth term $\Omega_x\{S_{n-j}\}$. Repeating this process $m-1$ times by putting $m = 2, 3, \ldots, m$ and solving the determinant each time, we finally get (6.3) .

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GENERALIZED NÖRLUND SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES .

By

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1. A series Σa_n with sequence of partial sums $\{S_n\}$ is said to be summable to S by the Nörlund $(N, P_n^{\alpha}, q_n^{\beta})$ - method ,

$$\alpha > -1, \beta > -1, \text{ if}$$

$$\lim_{n\to\infty} \quad t_n^{\alpha, \beta} \to S, \text{ as } n \to \infty$$

where

$$t_n^{\alpha,\beta} = \frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha} q_{n-k}^{\beta} \; S_{n-k} ,$$

$$P_n^{(\alpha)} = \sum_{k=0}^{n} p_k^{\alpha} ,$$

$$\Omega_n^{(\beta)} = \sum_{k=0}^{n} q_k^{\beta}$$

and

$$R_n^{(\alpha,\beta)} = \sum_{k=0}^{n} r_n^{\alpha,\beta} = \sum_{k=0}^{n} p_k^{\alpha} \,.\, q_{n-k}^{\beta} ,$$

$\{p_n\}$ and $\{q_n\}$ be the sequences of real numbers such that $p_n \geqslant 0$, $q_n \geqslant 0$ for all $n \geqslant 0$.

In the special case when $q_n = 1$ for all n, then $(N, p_n^{\alpha}, q_n^{\beta})$ method reduces to (N, p_n^{α}) - method (Cass [1]). Now if $\alpha = \beta = 1$, then $(N, p_n^{\alpha}, q_n^{\beta})$ - method reduces (N, p_n, q_n) - method and further if $q_n = 1$ for all n, then this method reduces to the familiar Nörlund (N, p_n) - method .

In this paper, we take $\{ p_n^{\alpha} \}$ and $\{ r_n^{\alpha,\beta} \}$ to be real, non-negative and monotonic non-increasing sequences such that $p_n^{(\alpha)} \to \infty$, $Q_n^{(\beta)} \to \infty$ and $R_n^{(\alpha,\beta)} \to \infty$, as $n \to \infty$, further we assume that $(N, p_n^{\alpha}, q_n^{\beta})$ is a regular method of summability.

2. Let $f(t)$ be a periodic function with period 2π and integrable L over $(-\pi, \pi)$. Let its Fourier series be given by

$$(2.1) \qquad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t) .$$

The conjugate series of (2.1) is

$$(2.2) \quad \sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) .$$

We shall use the following notations :

$$\phi(t) = f(x+t) + f(x-t) - 2 f(x) .$$

$$\psi(t) = f(x+t) - f(x-t),$$

$$\Phi(t) = \int_0^t |\phi(u)| \, du,$$

$$\Psi(t) = \int_0^t |\psi(u)| \, du,$$

$$N_n(t) = \frac{1}{2\pi R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha} q_{n-k}^{\beta} \frac{\sin(n-k+1/2)t}{\sin \frac{t}{2}},$$

$$N_n(t) = \frac{1}{2\pi R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha} q_{n-k}^{\beta} \frac{\cos(n-k+1/2)t}{\sin \frac{t}{2}},$$

$T = [1/t]$ where $[t]$ denotes the integral part of t.

3. Singh [3] has proved the following theorems :

Theorem A : If (N, p_n) be a regular Nörlund method, defined by a non-negative monotonic, non-increasing sequence $\{p_n\}$ such that $p_n \to \infty$ then if

$$(3.1) \quad \Phi(t) = o\left[\frac{p(1/t)}{p(1)/t}\right]$$

as $t \to +0$, the Fourier series of $f(t)$, at $t = x$ is summable (N, p_n) to $f(x)$.

Theorem B: If the sequence $\{p_n\}$ be same as in theorem A, then if

$$(3.2) \quad \Psi(t) = o\,\frac{p(1/t)}{P(1/t)},$$

as $t \to +0$, the conjugate series (2.2) is summable (N, p_n) to

$$\frac{1}{2\pi}\int_0^\pi \psi(t)\ \cot\frac{t}{2}\ dt$$

at every point, where this integral exists.

4. In the present paper we prove the following theorems :

Theorm 1. Let $\lambda(t)$ and $k(t)$ be two positive functions, if

$$(4.1) \quad \Phi(t) = \int_0^t |\phi(u)|\, du = o\left[\frac{\lambda(1/t)\, p_r^{\alpha}\, q_{n-r}^{\beta}}{K\left(R_t^{(\alpha,\beta)}\right)}\right]$$

as $t \to +0$.

and

$$(4.2) \quad \lambda(n)\, R_n^{(\alpha,\beta)} = o\left[K\left(R_n^{(\alpha,\beta)}\right)\right]$$

as $n \to \infty$, then the Fourier series of $f(t)$, at $t = x$ is summable $(N, p_n^{\alpha}, q_n^{\beta})$ to $f(x)$ where $\{p_n^{\alpha}\}$, $\{q_n^{\beta}\}$ and $\{r_n^{\alpha,\beta}\}$ are real, non-negative and monotonic non-increasing sequences such that

$$p_n^{(\alpha)} \to \infty \text{ and } R_n^{(\alpha,\beta)} \to \infty, \text{ as } n \to \infty.$$

Theorem 2. If the sequence $\{p_n^{\alpha}\}$, $\{q_n^{\beta}\}$ and the functions $\lambda(t)$ and $k(t)$ be same as in theorem 1, then if

$$(4.3)\quad \Psi(t) = \int_0^t |\psi(u)|\,du = o\left[\frac{\lambda(1/t)\; p_r^{\alpha}\, q_{n-r}^{\beta}}{K(R_t^{(\alpha,\beta)})}\right]$$

as $t \to +0$, then the conjugate (2.2) is summable $(N, p_n^{\alpha}, q_n^{\beta})$

$$\frac{1}{2\pi}\int_0^{\pi} \psi(t)\ \cot\ \frac{t}{2}\,dt$$

at every point where this integral exists .

5. The following lemmas are required for the proof of our theorems:

Lemma 1. If the sequence $\{p_n^{\alpha}\}$, $\{q_n^{\beta}\}$ and $\{r_n^{\alpha,\beta}\}$ the non-negative and non-increasing sequences, then for $0 \leqslant a \leqslant b \leqslant \infty$ and $0 \leqslant t \leqslant \pi$ and any n,

$$\left|\sum_{k=a}^{b} p_k^{\alpha}\, q_{n-k}^{\beta}\, e^{i(n-k)t}\right| \leqslant A\, R_r^{(\alpha,\beta)}$$

where A is an absolute constant .

The proof of Lemma follows on the lines of McFadden [2] .

Lemma 2. If $0 \leqslant t \leqslant \frac{1}{n}$, then $N_n(t) = o(n)$.

Proof . We have

$$N_n(t) = \frac{1}{2\pi\, R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha}\, q_{n-k}^{\beta}\ \frac{\sin(n-k+1/2)t}{\sin t/2}$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha}\, q_{n-1}^{\beta}\ \frac{(n-k+\frac{1}{2}\,t}{t/2}\right]$$

$$= o\left[\frac{n}{R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha}\, q_{n-k}^{\beta}\right]$$

$$= o(n).$$

Lemma 3. For $\frac{1}{n} \leqslant t \leqslant \delta \leqslant \pi$

$$N_n(t) = o\left[R_\tau^{(\alpha,\beta)} \,/\, R_n^{(\alpha,\beta)}\, t\right].$$

Proof. we have

$$|N_n^{(t)}| = \frac{1}{2\pi\, R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha}\, q_{n-k}^{\beta}\, \frac{\sin(n-k+\frac{1}{2})t}{\sin t/2} \Big|$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}\, t} \left| \operatorname{Img} e^{it/2} \sum_{k=0}^{n} p_k^{\alpha}\, q_{n-k}^{\beta} \exp(i(n-k)t) \right|\right]$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}\, t} \sum_{k=0}^{n} p_k^{\alpha}\, q_{n-k}^{\beta} \exp i(n-k)t \,\Big|\, \right]$$

$$o\left[\frac{R_\tau^{(\alpha,\beta)}}{R_n^{(\alpha,\beta)}}\right] \text{by Lemma 1}.$$

Lemma 4. If $\frac{1}{n} \leqslant t \leqslant \delta \leqslant \pi$ then

$$N_n(t) = o\left[\frac{R_\tau^{(\alpha,\beta)}}{R_n^{(\alpha,\beta)}\, t}\right].$$

The proof is similar to that of lemma 3.

6. Proof of theorem 1.

Let

$$S_n(x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} A_k(x).$$

Then we have

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^{x} \phi(t)\, \frac{\sin(n-k+\frac{1}{2})t}{\sin t/2}\, dt$$

Usiug (1.1), we get

$$t_n^{(\alpha,\beta)} - f(x) = \frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha}\, q_{n-k}^{\beta}\, [S_{n-k}(x) - f(x)]$$

$$= \frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha}\, q_{n-k}^{\beta}\, \frac{1}{2\pi} \int_0^{\pi} \phi(t)\, \frac{\sin(n-k+\frac{1}{2})t}{\sin t/2}$$

$$= \int_0^{\pi} \phi(t) \frac{1}{2\pi R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha} q_{n-k}^{\beta} \frac{\sin(n-k+\frac{1}{2})t}{\sin t/2} dt$$

$$= \int_0^{\pi} \phi(t) N_n(t) dt$$

In order to prove the theorem, we have to show that under our assumptions

$$\int_0^{\pi} \phi(t) N_n(t) dt = o(1), \text{ as } n\to\infty .$$

We write for $0 < \delta < \pi$

$$\int_0^{\pi} \phi(t) N_n(t) dt = \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right] \phi(t) N_n(t) dt$$

$$(6.1) \quad = I_1 + I_2 + I_3 \text{, say}$$

By Lemma 2, we have

$$I_1 = o\left[n \int_0^{1/n} |\phi(t)| dt \right.$$

$$= o\left[\frac{n\lambda(n) p_n^{\alpha} q_0^{\beta}}{K(R_n^{(\alpha,\beta)}} \right]$$

By assumption, that $\left\{ r_n^{(\alpha,\beta)} \right\}$ is non-negative, monotonic, non-incresaing sequence, we have obviously $np_n^{\alpha} q_0^{\beta} = n r_n^{\alpha,\beta} \leqslant R_n^{(\alpha,\beta)}$ and since $\lambda(n) N_n^{(\alpha,\beta)} = o\left[K(R_n^{(\alpha,\beta)}) \right]$, we have

$$(6.2) \quad I_1 = o(1), \text{ as } n\to\infty$$

Again by Lemma 3, we have

$$I_2 = o \int_{1/n}^{\delta} \phi(t) R_n(t) dt ,$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}} \int_{1/n}^{\delta} |\phi(t)| \frac{R_r^{(\alpha,\beta)}}{t} dt \right]$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}}\left\{\Phi(t)\,\frac{R_r^{(\alpha,\beta)}}{t}\right\}_{1/n}^{\delta}\right]$$

$$+ o\left[\int_{1/n}^{\delta}\frac{1}{R_n^{(\alpha,\beta)}}\,\Phi(1)\,\frac{R_r^{(\alpha,\beta)}}{t^2}\,dt\right]$$

$$+ o\left[\int_{1/n}^{\delta}\frac{1}{R_n^{(\alpha,\beta)}}\,\Phi(t)\,\frac{dR_r^{(\alpha,\beta)}}{t}\right]$$

$$(6.3) \qquad = I_{2\cdot 1} + I_{2\cdot 2} + I_{2\cdot 3}\text{, say}$$

$$I_{2\cdot 1} = o\left[\frac{1}{R_n^{(\alpha,\beta)}}\right] + o\left[\frac{1}{R_n^{(\alpha,\beta)}}\,\frac{\lambda(n)\,p_n^{\alpha}\,q_0^{\beta}}{K\left(R_n^{(\alpha,\beta)}\right)}\,nR_n^{(\alpha,\beta)}\right]$$

$$(6.4) \qquad = o(1),\ \text{as } n\to\infty\,.$$

Further

$$I_{2\cdot 2} = o\left[\frac{1}{R_n^{(\alpha,\beta)}}\int_{1/n}^{\delta}\Phi(t)\,\frac{R_r^{(\alpha,\beta)}}{t^2}\,dt\right]$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}}\int_{1/n}^{\delta}\frac{\lambda(1/t)\,p_r^{\alpha}\,q_{n-r}^{\beta}}{K\left(R_r^{(\alpha,\beta)}\right)}\,\frac{R_r^{(\alpha,\beta)}}{t^2}\,dt\right]$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}}\int_{1/\delta}^{n}\frac{\lambda(s)\,p_s^{\alpha}\,q_{n-s}^{\beta}}{K\left(R_s^{(\alpha,\beta)}\right)}\,R_s^{(\alpha,\beta)}\,ds\right]$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}}\int_{1/\delta}^{n}p_s^{\alpha}\,q_{n-s}^{\beta}\,ds\right]$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}}\sum_{k=0}^{n}p_k^{\alpha}\,q_{n-k}^{\beta}\right]$$

$$(6.5) \qquad = o(1),\ \text{as } n\to\infty\,.$$

Again we have

$$I_{2\cdot 3} = \frac{1}{R_n^{(\alpha,\beta)}} \int_{1/n}^{\delta} \phi\,(t)\frac{1}{t}\, d\, R_\tau^{(\alpha,\beta)}$$

$$= o\left[\frac{1}{R_n^{(\alpha,\beta)}} \int_{1/\delta}^{n} \phi\left(\frac{1}{u}\right)\ u\,dR_{(u)}^{(\alpha,\beta)}\right.$$

$$= o(1) + o\left[\frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=1}^{n-1} k\, p_k^{\alpha}\ q_{n-k}^{\beta}\ \Phi\left(\frac{1}{k}\right)\right]$$

$$= o\,(1) + o\left[\frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=1}^{n-1} \frac{R_k^{(\alpha,\beta)}\ \lambda\,(k)\ p_k^{\alpha}\ q_{n-k}^{\beta}}{K\,(R_k^{(\alpha,\beta)})}\right.$$

(6.6) $\qquad = o(1)$, as $n \to \infty$.

Combining (6.4), (6.5) and **(6.6)**, we have

(6.7) $\qquad I_2 = o(1)$.

Finally, since the method of summability is regular, we have

(6.8) $\quad I_3 = o(1)$, as $n \to \infty$

by Riemann Lebsegue theorem .

Hence on collecting (6.2), (6.7) and (6.8), this completes the proof of theorem 1 .

7. Proof of theorem 2 . Let $\bar{S}_n\,(x)$ denotes the n^{th} partial sum of series (2.2), we have

$$\bar{S}_n\,(x) = \frac{1}{2\pi}\int_0^{\pi} \psi\,(t)\ \frac{\cos t/2 - \cos\,(n+\frac{1}{2})t}{\sin t/2}\,dt$$

For $\Sigma B_n\,(x)$, making use of (1.1), we get

$$\bar{t}_n\,(\alpha,\beta) - \frac{1}{2\pi}\int_0^{\pi} \psi(t)\ \cos\frac{t}{2}\ \ dt$$

$$= \frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha}\ q_{n-k}^{\beta}\ \ s_{n-k}(x) - \frac{1}{2\pi}\int_0^{\pi} \psi\,(t)\ \cot\frac{t}{2}\ dt$$

$$= \frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha} \; q_{n-k}^{\beta} \; \frac{1}{2\pi} \int_0^{\pi} \psi(t) \; \frac{\cos t/2 \cos(n-k+\frac{1}{2})t}{\sin t/2} \, dt$$

$$- \frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\frac{t}{2} \, dt .$$

$$= - \int_0^{\pi} \psi(t) \; \frac{1}{2\pi \; R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha} \; q_{n-k}^{\beta} \; \frac{\cos (n-k+\frac{1}{2})t}{\sin t/2} \, dt$$

$$= - \int_0^{\pi} \psi(t) \, \bar{N}_n (t) \, dt \; .$$

In order to prove the theorem, we have to show that under our assumptions,

$$\int_0^{\pi} \psi(t) \, \bar{N}_n (t) \, dt = o(1), \text{ as } n \to \infty \; .$$

For $0 < \delta < \pi$, we have

$$\int_0^{\pi} \psi(t) \, \bar{N}_n (t) \, dt = \Big[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \Big] \psi(t) \, \bar{N}_n (t) \, dt .$$

(7.1) $\qquad = J_1 + J_2 + J_3$, Say.

Since the conjugate function exists, therefore

$$\frac{1}{2\pi} \int_0^{1/n} \psi(t) \; \cot \frac{t}{2} \; dt = o(1)$$

Hence

$$\frac{1}{2\pi} \int_0^{1/n} \psi(t) \; \cot \frac{t}{2} \; dt = J_1$$

$$= \frac{1}{2\pi} \int_0^{1/n} \psi(t) \Big[\cot \frac{t}{2} - \frac{1}{R_n^{(\alpha,\beta)}} \sum_{k=0}^{n} p_k^{\alpha} \; q_{n-1}^{\beta} \; \frac{\cos (n-k+\frac{1}{2})t}{\sin t/2} \Big] dt$$

$$= \frac{1}{2\pi \; R_n^{(\alpha,\beta)}} \int_0^{1/n} \psi(t) \sum_{k=0}^{n} p_k^{\alpha} \; q_{n-k}^{\beta} \Big\{ \sum_{u=0}^{n-k} 2 \sin u \, t \Big\} \, dt .$$

$$= \Big[\frac{1}{R_n^{(\alpha,\beta)}} \int_0^{1/n} | \psi(t) | \sum_{k=0}^{n} n \, p_k^{\alpha} \; q_{n-k}^{\beta} \, dt$$

$$= o \left[n \int_0^{1/n} | \psi(t) | \, dt \right.$$

$$= o \left[\frac{n \, \lambda(n) \, p_n^{\alpha} \, q_0^{\beta}}{K \left(R_n^{(\alpha,\beta)} \right)} \right]$$

(7.2) $\qquad = o(1)$

Therefore

$$J_1 = o(1)$$

For $\frac{1}{n} \leqslant t \leqslant \delta$

$$J_2 = o \left[\int_{1/n}^{\delta} | \psi(t) | \, | \bar{N}_n(t) | \, dt \right.$$

$$= o \left[\int_{1/n}^{\delta} | \psi(t) | \, \frac{R_{\tau}^{(\alpha,\beta)}}{R_n^{(\alpha,\beta)} \, t} \, dt \right]$$

(7.3) $\qquad = o(1)$, *as in* J_2.

Also

(7.4) $\qquad J_3 = o(1),$

by virtue of Riemann Lebesgue theorem and regulartity of method of sumation.

This completes the proof of theorem 2.

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INTEGRAL REPRESENTATIONS OF MULTIPLE BASIC HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

Certain integral representations of multiple basic hypergeometric function of Srivastava are obtained by the application of certain operators ot fractional q-differentiation. The results derived in this paper generalize the work of the authors [7] and Upadhyay [11]. Some interesting special cases are also given.

1. INTRODUCTION

Al-Salam [3 and Agrawal [2] defined the operator of fractional q-differentiation in the form

$$(1.1)\quad D^{\alpha}_{q,x} f(x) = (1-q)^{-\alpha-1}\,\Pi\left[\begin{matrix}-\alpha;\\ 1;\end{matrix}\,q\right]\int_0^x [x-t_q]_{-\alpha-1} f(t)\,d(t;q),$$

where the basic integral is defined through the following relation due to Hahn [6] :

$$(1.2)\quad \int_0^x f(t)\,d(t;q) = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k).$$

It can be seen that (1.1) is equivalent to

$$(1\ 3)\quad D^{\alpha}_{q,x} f(x) = (1-q)^{-\alpha} x^{-\alpha} \sum_{n\geqslant 0} \frac{(q^{-\alpha};q)_n\, q^n f(xq^n)}{(q;q)_n}$$

so that

$$(1.4)\quad D^{\alpha}_{q,x} x^{\mu-1} = (1-q)^{-\alpha}\,\Pi\left[\begin{matrix}\mu-\alpha;\\ \mu;\end{matrix}\,q\right] x^{\mu-\alpha-1},$$

where for $|q| < 1$, further let

(1.5) $$(a;q)_n = \prod_{j=0}^{\infty} \frac{(1-aq^j)}{(1-aq^{n+j})}$$

for arbitrary a and n.

We also have

(1.6) $$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1});\quad n \geqslant 1 .$$

The generalized basic hypergeometric series is defined as,

(1.7) $${}_A\phi_B\left[\begin{matrix}(a);\\(b);\end{matrix} q;x\right] = {}_A\phi_B\left[\begin{matrix}a_1,\dots,a_A;\\b_1\dots,b_B;\end{matrix} q;\ x\right]$$

$$= \sum_{n\geqslant 0} \frac{(a_1;q)_n \dots (a_A;q)_n\, x^n}{(b_1;q)_n \dots (b_B;q)_n (q;q)_n}$$

where, $|q| < 1$ snd $|x| < 1$.

Also, the multiple basic hypergeometric function introduced by Srivastava [11, *p.* 318, *Eq.* (3.1)] (Also see [9, *p.* 350, *Eq* (284)] is defined as :

(1.8) $$\phi^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}] : [(b'):\phi'];\dots;\\ [(c):\psi',\dots,\psi^{(n)}] : [(d'):\delta'];\dots;\end{matrix}\ \begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ q;\ x_1,\dots,x_n\right)$$

$$= \sum_{m_1,\dots,m_n\geqslant 0} \frac{\prod_{j=1}^{A}(a_j;q)_{m_1\theta'_j+\dots+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j;q)_{m_1\psi'_j+\dots+m_n\psi_j^{(n)}}}\ \frac{\prod_{j=1}^{B'}(b'_j;q)_{m_1\phi'_j}\dots}{\prod_{j=1}^{D'}(d'_j;q)_{m_1\delta'_j}\dots}$$

$$\cdot\ \frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)};q)_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)};q)_{m_n\delta_j^{(n)}}}\ \frac{x_1^{m_1}}{(q;q)_{m_1}}\dots\frac{x_n^{m_n}}{(q;q)_{m_n}},$$

where the arguments $x_1,\dots,x_n$, the complex parameters

$$\left\{\begin{matrix} a_j,\ j=1,\dots,A;\ b_j^{(k)},\ j=1,\dots,B_j^{(k)}\ ;\\ c_j,\ j=1,\dots,C;\ d_j^{(k)}\ j=1,\dots,\ D_j^{(k)}\ ;\ k=1,\dots,n\end{matrix}\right.$$

and the associated coeficients

$$\left\{ \begin{array}{l} \theta_j^{(k)}, \ j=1,\ldots,A; \ \phi_j^{(k)}, \ j=1,\ldots,B_j^{(k)}; \\ \psi_j^{(k)}, \ j=1,\ldots,D; \ \delta_i^{(k)}, \ j=1,\ldots,D_j^{(k)}; \ k=1,\ldots,n \end{array} \right.$$

are so constrained that the multiple series (1.8) converge.

The generalized basic Lauricella functions, which follows from the definition (1.8) are defined as: (cf. Agrawal [1])

$$(1.9) \quad \phi_A^{(n)}\left[\alpha,\beta_1,\ldots,\beta_n;\ \gamma_1,\ldots,\gamma_n;\ q;\ x_1,\ldots x_n\right]$$

$$= \sum_{m_1,\ldots,m_n \geqslant 0} (\alpha;q)_{m_1+\ldots+m_n} \prod_{j=1}^{n} \left\{ \frac{(\beta_j;q)_{m_j} x_j^{m_j}}{(\gamma_j;q)_{m_j} (q;q)_{m_j}} \right\},$$

$$(1.10) \quad \phi_B^{(n)}\left[\alpha_1,\ldots,\alpha_n,\ \beta_1,\ldots,\beta_n;\ \gamma;\ q;\ x_1,\ldots,x_n\right]$$

$$= \sum_{m_1,\ldots,m_n \geqslant 0} \frac{1}{(\gamma;q)_{m_1+\ldots+m_n}} \prod_{j=1}^{n} \left\{ \frac{(\alpha_j;q)_{m_j} (\beta_j;q)_{m_j} x_j^{m_j}}{(q;q)_{m_j}} \right\},$$

$$(1.11) \quad \phi_C^{(n)}\left[\alpha,\beta;\ \gamma_1,\ldots,\gamma_n:\ q;\ x_1,\ldots,x_n\right]$$

$$= \sum_{m_1,\ldots,m_n \geqslant 0} (\alpha;q)_{m_1+\ldots+m_n} (\beta;q)_{m_1+\ldots+m_n} \prod_{j=1}^{n} \left\{ \frac{x_j^{m_j}}{(\gamma_j;q)_{m_j} (q;q)_{m_j}} \right\},$$

$$(1.12) \quad \phi_D^{(n)}\left[\alpha,\beta_1,\ldots,\beta_n;\ \gamma;\ q;\ x_1,\ldots,x_n\right]$$

$$= \sum_{m_1,\ldots,m_n \geqslant 0} \frac{(\alpha;q)_{m_1+\ldots+m_n}}{(\gamma;q)_{m_1+\ldots+m_n}} \prod_{j=1}^{n} \left\{ \frac{(\beta_j;q)_{m_j} x_j^{m_j}}{(q;q)_{m_j}} \right\}$$

where $|x_j| < 1$; $j=1,\ldots,n$ and $|q| < 1$.

Out of the above four functions, $\phi_D^{(n)}$ was introduced and studied earlier by Andrews [4] .

The following results will be needed in the analysis that follows:

$$(1.13) \qquad {}_1\phi_0\left[a; -, q; x\right] = \Pi\left[\begin{matrix} ax; \\ x\ ; \end{matrix}\ q\right],$$

$$(1.14) \qquad [x-y]_v = x^v\,[1-y/x]_v = \prod_{n=0}^{\infty} \frac{(1-yq^n/x)}{(1-yq^{v+n}/x)}\ .$$

$$(1.15) \qquad \Pi\left[\begin{matrix} a_1,\dots,a_A; \\ d_1,\dots,b_B; \end{matrix}\ q\right] = \prod_{j=0}^{\infty} \frac{(1-a_1q^j)\ \dots\ (1-a_Aq^j)}{(1-b_1q^j)\ \dots\ (1-b_Bq^j)}\ .$$

The notation $D_{q,x}^{b,\alpha-1}\Big[\ f(x)\ \Big]$ will be used to denote $D_{q,x}^{b}\Big[x^{\alpha-1} f(x)\Big]$.

2. INTEGARL REPRESENTATION

The following integral representations are to be established here:

$$(2.1) \qquad \phi^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}] : [(b') : \phi'] ; \dots; \\ [(c):\psi',\dots,\psi^{(n)}] : [(d') : \delta'] ; \dots; \end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}) : \phi^{(n)}]; \\ [(d^{(n)}) : \delta^{(n)}];\end{matrix}\ q;\ x_1,\dots,x_n\right)$$

$$= \frac{1}{(1-q)^n}\,\Pi\left[\begin{matrix} b_1-d_1,\dots,b_n-d_n,d_1,\dots,d_n; \\ b_1\ ,\dots,\ b_n\ ,\ 1\ ,\dots,\ 1; \end{matrix}\ q\right] \int_0^1 \dots \int_0^1 u_1^{\,d_1-1}\ \dots\ u_n^{\,d_n-1}$$

$$[1-u_1q]_{b_1-d_1-1} \dots [1-u_nq]_{b_n-d_n-1}\ \phi^{A:B'+1;\dots;B^{(n)}+1}_{C:D'+1;\dots;D^{(n)}+1}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]: \\ [(c):\psi',\dots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'],\ [b_1:1];\dots;\ [(b^{(n)}):\phi^{(n)}],\ [b_n:1]; \\ [(d'):\delta'],\ [d_1:1];\dots;\ [(d^{(n)}):\delta^{(n)}],\ [d_n:1];\end{matrix}\ q; u_1x_1,\dots,u_nx_n\right)$$

$$.\ d(u_1;q)\dots d(u_n;q)$$

provided that $Re\,(d_j) > 0,\ |x_j| < 1;\ j = 1,..n$.

$$(2.2) \qquad \phi^{A:B;\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots\theta^{(n)}]:\ [(b'):\phi'];\dots; \\ [(c):\psi',\dots,\psi^{(n)}]:\ [(d'):\delta'];\dots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}]; \\ [(d^{(n)});\delta^{(n)}];\end{matrix}\ q;\ x_1x_2,\dots,x_1x_n,x_1\ \right)$$

$$= \frac{1}{(1-q)^n} \Pi\left[\begin{matrix} e-f, b_1-d_1, \ldots, b_{n-1}-d_{n-1}, f, d_1, \ldots, d_{n-1}; \\ 1, \ 1, \ \ldots, \ 1, \ e, \ b_1, \ldots, b_{n-1}; \end{matrix} q\right]$$

$$\int_0^1 \cdots \int_0^1 u_1^{f-1} u_2^{d_1-1} \cdots u_n^{d_{n-1}-1} [1-u_1q]_{e-f-1} [1-u_2q]_{b_1-d_1-1}$$

$$\cdots [1-u_nq]_{b_{n-1}-d_{n-1}-1}$$

$$\phi^{A+1:B'+1;\ldots;B^{(n-1)}+1;B^{(n)}}_{C+1:D'+1;\ldots;D^{(n-1)}+1;D^{(n)}} \left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], [e:1,\ldots,1]: \\ [(c):\psi',\ldots,\psi^{(n)}], [f:1,\ldots,1]:\end{matrix}\right.$$

$$\begin{matrix}[(b'):\phi'], [b_1:1];\ldots;[(b^{(n-1)}):\phi^{(n-1)}], [b_{n-1}:1]; \\ [(d'):\delta'], [d_1:1];\ldots;[(d^{(n-1)}):\delta^{(n-1)}], [d_{n-1}:1];\end{matrix}$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}]; \\ [(d^{(n)}):\delta^{(n)}];\end{matrix} q; u_1u_2x_1x_2, \ldots, u_1u_nx_1x_n, u_1x_1\right)$$

$$\cdot\, d(u_1;q) \ldots d(u_n;q)$$

provided that $Re(f) > 0$, $Re(d_j) > 0$, $j = 1,\ldots,(n-1)$;

$|x_i| < 1$; $i = 1,\ldots,n$.

$$(2.3) \quad \phi^{A+1:B';\ldots;B^{(n)}}_{C+1:D';\ldots;D^{(n)}} \left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], [e:1,\ldots,1]: \\ [(c):\psi',\ldots,\psi^{(n)}], [f:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi']; \ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix} q; ux_1,\ldots,ux_n\right)$$

$$= \frac{1}{(1-q)} \Pi\left[\begin{matrix} f-e, e; \\ f, 1 \end{matrix} \ q\right] \int_0^1 t^{e-1} [1-tq]_{f-e-1}$$

$$\cdot\, \phi^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}} \left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: [(b'):\phi'];\ldots; \\ [(c):\psi',\ldots,\psi^{(n)}]: [(d'):\delta'];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}]; \\ [(d^{(n)}):\delta^{(n)}];\end{matrix} q; utx_1,\ldots,utx_n\right) d(t;q)$$

provided that $Re(e) > 0$, $|ux_j| < 1$; $j = 1,\ldots,n$.

Proof of (2.1). To prove (2.1), we start with

$$D^{d_1-b_1,\, d_1-1}_{q,\, x_1} \cdots D^{d_n-b_n,\, d_n-1}_{q,\, x_n} \left\{ \phi^{A:B'+1;\ldots B^{(n)}+1}_{C:D'+1;\ldots D^{(n)}+1} \left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: \\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}[(b'):\phi'], [b_1:1];\ldots;[(b^{(n)}):\phi^{(n)}], [b_n:1]; \\ [(d'):\delta'], [d_1:1];\ldots;[(d^{(n)}):\delta^{(n)}], [d_n:1];\end{matrix} q; x_1,\ldots,x_n\right)\right\}$$

on using (1.7), it gives

$$D_{q,\,x_1}^{d_1-b_1}\cdots D_{q,\,x_n}^{d_n-b_n}\left\{\sum_{m_1,\ldots,m_n\geqslant 0}\frac{\prod_{j=1}^{A}(a_j;q)_{m_1\theta'_j+\ldots+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j;q)_{m_1\psi'_j+\ldots+m_n\psi_j^{(n)}}}\right.$$

$$\frac{\prod_{j=1}^{B'}(b'_j;q)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)};q)_{m_n\phi_j^{(n)}}\prod_{j=1}^{n}(b_j;q)_{m_j}}{\prod_{j=1}^{D'}(d'_j;q)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)};q)_{m_n\delta_j^{(n)}}\prod_{j=1}^{n}(d_j;q)_{m_j}}$$

$$\left.\frac{x_1^{m_1+d_1-1}}{(q;q)_{m_1}}\cdots\frac{x_n^{m_n-d_n-1}}{(q;q)_{m_n}}\right\}.$$

Applying (1.4) and (1.1), we find that

$$(2.4)\quad x_1^{b_1-1}\ldots x_n^{b_n-1}\Pi\begin{bmatrix}b_1,\ldots,b_n; & q\\ d_1,\ldots,d_n; & \end{bmatrix}$$

$$\phi_{C:D';\ldots D^{(n)}}^{A:B\,;\ldots;B^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\,q;\,x_1,\ldots,x_n\right)$$

$$=\frac{1}{(1-q)^n}\,\Pi\begin{bmatrix}b_1-d_1,\ldots,b_n-d_n; & q\\ 1,\ldots,1 & ;\end{bmatrix}\int_0^{x_1}\cdots\int_0^{x_n}u_1^{d_1-1}\ldots u_n^{d_n-1}$$

$$\cdot\,[x_1-u_1q]_{b_1-d_1-1}\cdots[x_n-u_nq]_{b_n-d_n-1}$$

$$\cdot\quad\phi_{C:D'+1;\ldots;D^{(n)}+1}^{A:B'+1;\ldots;B^{(n)}+1}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: & [(b'):\phi'],[b_1:1];\ldots;\\ [(c):\psi',\ldots,\psi^{(n)}]: & [(d'):\delta'],[d_1:1];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}],[b_n:1];\\ [(d^{(n)}):\delta^{(n)}],[d_n:1];\end{matrix}\,q;u_1,\ldots,u_n\right)d(u_1;q)\ldots d(u_n;q).$$

On using (1.14) and transforming u_i to u_ix_i for $i=1,\ldots,n$, the desired result (2.1) is obtained.

The remaining results (2.2) and (2.3) can then be proved in the same way.

3. SPECIAL CASES

On account of the most general character of the basic multiple hypergeometric function studied in this paper, numerous special cases associated with the generalized basic hypergeometric functions, their products, basic Appell functions and generalized Lanricella functions etc. can be derived. But for sake of brevity they are not presented here. However we give here some interesting results associated with the basic Lanricella functions and product of generalized basic hypergeometric funtions:

If we take $B' = \dots = B^{(n)} = D' = \dots = D^{(n)} = C = 0$, $A = 1$, $\theta' = \dots = \theta^{(n)} = 1$; in (2.1) and then making the use of (1.13), we obtain an interesting result

$$(3.1) \quad \frac{1}{(1-q)^n} \int_0^1 \dots \int_0^1 u_1^{d_1-1} \dots u_n^{d_n-1} [1-u_1 q]_{b_1-d_1-1} \dots [1-u_n q]_{b_n-d_n-1} \; \phi_A^{(n)} \Big[a_1, b_1, \dots, b_n; d_1, \dots, d \,; q; u_1 x_1, \dots, u_n x_n \Big] d(u_1;q) \dots d(u_n;q)$$

$$= \Pi \left[\begin{matrix} a_1(x_1 + \dots + x_n), b_1, \dots, b_n, 1, \dots, 1 \,; \\ (x_1 + \dots + x_n), b_1-d_1, \dots, b_n-d_n, d_1, \dots, d_n; \end{matrix} \; q \right]$$

provided that $Re(d_j) > 0$, $|x_j| < 1$; $j = 1, \dots n$.

For $A = D = \dots = D^{(n)} = 0$, $C = 1$, $B' = \dots = B^{(n)} = 2$, $\psi' = \dots = \psi^{(n)} = \phi' = \dots = \phi^{(n)} = 1$; $b'_1 = b'$, $b''_1 = b''$, $\dots$, $b^{(n)}_1 = b^{(n)}$ and $b'_2 = d_1$, $b''_2 = d_2, \dots, b^{(n)}_2 = d_n$; (2.1) reduces to the following result:

$$(3.2) \quad \phi^{0:2;\dots;2}_{1:0;\dots;0} \left(\begin{matrix} - & :[b':1], [d_1:1]; \dots; [b^{(n)}:1], d_n:1]; \\ [c_1:1,\dots,1]: & - \qquad ; \dots ; \qquad - \qquad ; \end{matrix} \; q; x_1, \dots, x_n \right)$$

$$= \frac{1}{(1-q)^n} \Pi\begin{bmatrix} b_1-d_1,\ldots,b_n-d_n, & d_1,\ldots,d_n; \\ b_1,\ldots,b_n & , 1,\ldots,1; \end{bmatrix}q \int_0^1 \cdots \int_0^1 u_1^{d_1-1} \cdots u_n^{d_n-1}$$

$$[1-u_1q]_{b_1-d_1-1} \cdots [1-u_nq]_{b_n\ d_n-1}$$

$$\phi_B^{(n)}\left[b',\ldots,b^{(n)}, b_1,\ldots,b_n; c_1;q; u_1x_1,\ldots,u_nx_n\right] d(u_1;q) \ldots d(u_n;q)$$

provided that $Re(d_j) > 0$, $|x_j| < 1$; $j = 1,\ldots,n$.

If we take $A = 2$; $D' = \ldots = D^{(n)} = 1$,

$d_1' = b_1,\ldots, d_1^{(n)} = b_n$; $B' = \ldots = B^{(n)} = 0 = C$;

$\theta' = \ldots = \theta^{(n)} = \delta' = \ldots = \delta^{(n)} = 1$; in (2.1) then we obtain

$$(3.3) \quad \phi^{2:0;\ldots;0}_{0:1;\ldots;1}\left(\begin{matrix}[a_1:1,\ldots,1], [a_2:1,\ldots,1]: - ; & \ldots; & - ; \\ - \qquad : [b_1:1]; & \ldots; & [b_n:1]; \end{matrix}\ q; x_1,\ldots,x_n\right)$$

$$= \frac{1}{(1-q)^n} \Pi\begin{bmatrix} b_1-d_1,\ldots,b_n-d_n, & d_1,\ldots,d_n; \\ b_1,\ldots,b_n & , 1,\ldots,1; \end{bmatrix}q \int_0^1 \cdots \int_0^1 u_1^{d_1-1} \cdots u_n^{d_n-1}$$

$$[1-u_1q]_{b_1-d_1-1} \cdots [1-u_nq]_{b_n-d_n-1}\ \phi_C^{(n)}\Big[a_1, a_2; d_1,\ldots,d_n;$$

$$q; u_1x_1,\ldots,u_nx_n\Big]\ d(u_1;q) \ldots d(u_n;q) \ ,$$

provided that $Re(d_j) > 0$, $|x_j| < 1$, $j = 1,\ldots,n$.

Also if we take $A = C = B' = B^{(n)} = 1$; $b_1' = d_1,\ldots, b_1^{(n)} = d_n$;

$D' = \ldots = D^{(n)} = 0$; $\theta' = \ldots \theta^{(n)} = \psi' = \ldots = \psi^{(n)} = 1$,

$\phi' = \ldots = \phi^{(n)} = 1$; in **(2.1)** and then making use of the known result, Andrews [4, **(4.1)**], namely

$$(3.4) \quad \phi_D^{(n)}\Big[a, b_1,\ldots,b_n; c; q; x_1,\ldots,x_n\Big]$$

$$= \Pi\begin{bmatrix} a, b_1x_1,\ldots,b_nx_n; \\ c, x_1,\ldots,x_n\ ; \end{bmatrix} q \ {}_{n+1}\phi_n \begin{bmatrix} c/a, x_1,\ldots,x_n; \\ b_1x_1,\ldots,b_nx_n; \end{bmatrix} q; a$$

then we obtain another interesting result

$$(3.5)\quad \phi^{1:1;\ldots;1}_{1:0;\ldots;0}\left(\begin{matrix}[a_1:1,\ldots,1]: & [d_1:1];\ldots; & [d_n:1]; \\ [c_1:1,\ldots,1]: & - \;\;;\ldots; & - \;\;;\end{matrix}\; q;\, x_1,\ldots,x_n\right)$$

$$= \frac{1}{(1-q)^n}\,\Pi\left[\begin{matrix}a_1, b_1-d_1,\ldots,b_n-d_n, d_1,\ldots,d_n; \\ c_1, b_1, \quad \ldots, b_n \quad, 1,\ldots,1;\end{matrix}\; q\right]\int_0^1\cdots\int_0^1 u_1^{d_1-1}\ldots u_n^{d_n-1}$$

$$[1-u_1q]_{b_1-d_1-1}\cdots[1-u_nq]_{b_n-d_n-1}\,\Pi\left[\begin{matrix}b_1u_1x_1,\ldots,b_nu_nx_n; \\ u_1x_1 \quad,\ldots, u_nx_n \;;\end{matrix}\; q\right]$$

$$\cdot\; {}_{n+1}\phi_n\left[\begin{matrix}c_1/a_1,\, u_1x_1,\ldots,u_nx_n\;; \\ b_1u_1x_1\;,\ldots,\; b_nu_nx_n\;;\end{matrix}\; q;a_1\right] d(u_1;q)\ldots d(u_n;q)$$

provided that $Re(d_j) > 0,\ |x_j| < 1,\ j = 1, \ldots n$.

For $A = C = 0,\ \phi' = \ldots = \phi^{(n)} = \delta' = \ldots = \delta^{(n)} = 1$; (2.1) then reduces to yet another interesting result associated with products of n generalized basic hypergeometric functions:

$$(3.6)\quad {}_{B'}\phi_{D'}\left[\begin{matrix}(b'); \\ (d');\end{matrix}\; q;x_1\right]\cdots{}_{B^{(n)}}\phi_{D^{(n)}}\left[\begin{matrix}(b^{(n)}); \\ (d^{(n)});\end{matrix}\; q;x_1\right]$$

$$= \frac{1}{(1-q)^n}\,\Pi\left[\begin{matrix}b_1\; d_1,\ldots,b_n-d_n; d_1,\ldots,d_n; \\ b_1 \quad,\ldots,\; b_n \quad, 1\;,\ldots,\;1;\end{matrix}\; q\right]\int_0^1\cdots\int_0^1 u_1^{d_1-1}\ldots u_n^{d_n-1}$$

$$\cdot\;[1-u_1q]_{b_1-d_1-1}\cdots[1-u_nq]_{b_n-d_n-1}\;{}_{B'+1}\phi_{D'+1}\left[\begin{matrix}(b'),b_1; \\ (d'),d_1;\end{matrix}\; q;u_1x_1\right]$$

$$\cdots\;{}_{B^{(n)}+1}\phi_{D^{(n)}+1}\left[\begin{matrix}(b^{(n)}),b_n; \\ (d^{(n)}),d_n;\end{matrix}\; q;u_nx_n\right] d(u_1;q)\ldots d(u_n;q)$$

provided that $Re(d_j) > 0,\ |x_j| < 1;\ j = 1,\ldots,n$.

A number of simlar results can be obtained from (2.2) and (2.3).

Finally, for $n = 2$, (2.1) to (2.3) reduce to the results given by the authors [7].

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VISCO-ELASTIC BOUNDARY LAYER FLOW PAST A STRETCHING PLATE WITH SUCTION AND HEAT TRANSFER

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ABSTRACT

In this paper, we have studied the Nusselt number for the convective heat transfer for leminar boundary layer flow of 'Walters liquid B' past a stretching plate with constant suction and with uniform temperature T. Further, we have obtained the temperature profile in the case of plate temperature $T=T_{\infty}+Ax^n$. In each case, the Nusselt number has been derived for Prandtl number $p<<1$, $p=1$ and $p>>1$.

1. INTRODUCTION

In day to-day problems, we come across the problems of flow past a non-rigid plate or boundary where suction is also present. The study of various aspects of suction is very useful in the design of aircraft wings. In more recent times one of the applications of suction is to reduse drag. In 1904, Prandtl considered suction in boundary layer control to-prevent separation. Siddappa and Khapate [6] extended the study Crane [2] to the visco-elastic flow past a stretching plate. In 1980, Siddappa and Hiremath [7] studied the problem of Siddappa and Khapate [6] with suction. Siddappa and Abel [5] worked on the boundary layer flow of Walters liquid B past a stretching plate with suction and heat transfer.

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In the present paper, the problem of Siddappa and Abel [5] is extended to study the Nusselt number for convective heat transfer.

2. Formation of the Problem

Let X-axis be along the plate in the direction of fluid motion, Y-axis be perpendicular to the plate and the slit be at origin. The boundary layer equation governing the flow of visco-elastic fluid (Walters liquid B) ([1], [4]) is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2} - K^*_0\left\{\frac{\partial^3 u}{\partial t\partial y^2} + u\frac{\partial^3 u}{\partial x\partial y^2}\right.$$

$$\left. + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right\} \qquad (2.1)$$

In the case of steady state boundary layer flow past a stretching plate with suction for the short memory approximation of Walters liquid B, the equation of motion and equation of continuity are [5];

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial^2 y} - K^*_0\left\{u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right.$$

$$\left. - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right\} \qquad (2.2)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (2.3)$$

respectively .

Boundary conditions are:

$$y = 0\,;\; u = mx,\;\; v = -\,v_0$$

$$y \to \infty\,;\; u = 0,\;\; v = -\,C,\;\; C > 0\,,\;\; u_y = 0 \qquad (2.4)$$

3. Solution of the Problem

Introducing dimensionless numbers

$$y' = \frac{y}{h}, \; u' = \frac{uh}{\nu}, \; x' = \frac{x}{h}$$

$$v' = \frac{vh}{\nu} \quad K_0^{*'} = \frac{k_0^*}{h^2}$$

and substituting all these in equations (2.2) — (2 4), we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - K_0^* \left\{ u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} \right\} \tag{3.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.2}$$

and boundary conditions:

$$\begin{aligned} &y = 0, \; u = mx, \; v = -v_0 \\ &y \to \infty, \; u = 0, \; v = -C, \; C > 0, \; u_y = 0 \end{aligned} \tag{3.3}$$

where dash has been suppressed for the convenience.

Setting the similarity solution of the form

$$u = mxf'(y), \; m > 0 \tag{3.4}$$

we get

$$v = -\{mf(y) + v_0\} \tag{3.5}$$

Putting (3.4) and (3.5) in (3.1), we get

$$f'^2(y) = f''(y)\,f(y) - \frac{v_0}{m}f''(y) = \frac{1}{m}f'''(y) - K_0^* \{2f'(y)\,f'''(y) - f(y)\,f^{iv}(y) - \frac{v_0}{m}f^{iv}(y) - f''^2(y)\} \tag{3.6}$$

and boundary conditions:

$$\begin{aligned} &y = 0; \; f = f' = 1 \\ &y \to \infty; \; C = mf + v_0, \; f'' = 0 \end{aligned} \tag{3.7}$$

Trying the solution assuming

$f'(y) = e^{-ry}, r>0$

we get r as the root of cubic equation

$$\frac{v_0}{m} \underset{0}{K^*} r^3 - \left(\frac{1}{m} - \underset{0}{K^*}\right) r^2 + \frac{v_0}{m} r + 1 = 0 \tag{3.8}$$

Hence the velocity components are

$$u = mx\, e^{-ry} \tag{3.9}$$

and

$$v = -\frac{m}{r}(1-e^{-ry})-v_0 \tag{3.10}$$

4. Heat Conduction in Linear Stretching Plate

The boundary layer energy equation is

$$\rho c_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = K\frac{\partial^2 T}{\partial y^2} \tag{4.1}$$

and boundary conditions:

$$y = 0;\ T = T_p$$
$$y \to \infty;\ T = T_s \tag{4.2}$$

where T_p is plate temperature and T_s is surrounding temperature.

Introducing dimensionless variables

$$\eta = ry \text{ and } \theta = \frac{T-T_s}{T_p-T_s} \tag{4.3}$$

the equation (4.1) becomes

$$\left[-\frac{m}{r}(1-e^{-ry})-v_0\right]\frac{\partial\theta}{\partial\eta} = \frac{r}{p}\frac{\partial^2\theta}{\partial\eta^2} \tag{4.4}$$

and boundary conditions reduce to

$$\eta = 0;\ \theta = 1$$
$$\eta \to \infty;\ \theta = 0 \tag{4.5}$$

The solution of equation (4.4) under the boundary condition (4.5) is

$$\theta(\eta) = \frac{\int_{\eta}^{\infty} exp \left\{ - \frac{m}{r^2} P(\eta + e^{-\eta} + \frac{v_0 r}{m} \eta) \right\} d\eta}{\int_{\eta}^{\infty} exp \left\{ - \frac{m}{r^2} P(\eta + e^{-\eta} + \frac{v_0 r}{m} \ ) \right\} d\eta} \tag{4.6}$$

Putting $\frac{m}{r^2} Pe^{-\eta} = t$, we have

$$\theta(\eta) = \frac{\gamma(\frac{p}{r}(m+v_0 r) \, , \, \frac{m}{r^2} Pe^{-\eta})}{\gamma(\frac{P}{r^2}(m+v_0 r) \, , \, \frac{m}{r^2} P)} \tag{4.7}$$

For $P = 1$, $\theta(\eta)$ becomes

$$\theta(\eta) = \frac{\gamma(\frac{1}{r^2}(m+v_0 r), \, \frac{m}{r^2} e^{-\eta})}{\gamma(\frac{1}{r^2}(m+v_0 r) \, , \, \frac{m}{r^2} \ )} \tag{4.8}$$

Differentiating (4.7) with respect to η, we get

$$\left.\frac{\partial\theta}{\partial\eta}\right|_0 = \frac{-(\frac{mp}{r^2} \ )^{1/r^2(m + v_0 r)p} \, exp(-\frac{m}{r^2} P)}{\gamma(\frac{P}{r^2}(m+v_0 r) \, , \, \frac{m}{r^2} P)} \tag{4.9}$$

The Nusselt Number

The Nusselt number $Nu(x)$ is defined by

$$\begin{aligned} Nu(x) &= -(\frac{\partial T}{\partial y})_{y=0} / T_p - T_s \\ &= -(\frac{\partial\theta}{\partial\eta}).(\frac{\partial\eta}{\partial y})_0 \end{aligned} \tag{4.10}$$

Hence, for $p=1$, the Nusselt number

$$Nu(x) = \frac{\left(\frac{m}{r^2}\right)^{1/r^2(m+v_0r)} exp\left(-\frac{m}{r^2}\right)}{\gamma\left(\frac{1}{r^2}(m+v_0r),\ \frac{m}{r^2}\right)} \tag{4.11}$$

For $P << 1$, using series expansion for the incomplete Gamma function,

$$\gamma(x,a) \doteq a^{x} \sum_{n=0}^{\infty} \frac{(-1)^n a^n}{(x+n)n!},$$

we have

$$Nu(x) \doteq \frac{m}{r} p \text{ for small } p \tag{4.12}$$

For $P >> 1$, applying stirling approximation formula for large p, we obtain

$$Nu(x) = \frac{\left(\frac{m}{r^2}p\right)^{p/r^2(m+v_0r)} exp\left(\frac{p}{r}v_0\right)}{(2\pi)^{1/2}\left(\frac{p}{r^2}(m+v_0r)\right)^{p/r^2(m+v_0r)-\frac{1}{2}}} \tag{4.13}$$

5. Heat Conduction for Linear Stretching Plate with Temperature $T_p = T_s + Ax^n$

Recalling the similarity variables $\eta = ry$ and defining the dimensionless temperature by $\frac{T-T_s}{T_s} = Ax^n g(\eta)$ and substituting in the boundary layer equation of temperature (4.1), we have

$$\frac{m}{r^2} p \left[\eta e^{-\eta} g(\eta) - \left(1-e^{-\eta}+\frac{v_0r}{m}\right) g'(\eta)\right] = g''(\eta) \tag{5.1}$$

and boundary condition:

$$\begin{aligned} \eta &= 0;\ g = 1 \\ \eta &= \infty;\ g = 0 \end{aligned} \tag{5.2}$$

Special case:

The solution of the equation (5.11) for $p = 1,\ \eta = 0$ subject to the boundary conditions (5.2) is

$$g(\eta) = \frac{\gamma(\frac{1}{r^2}(m+v_0 r),\ \frac{m}{r^2}e^{-\eta})}{\gamma(\frac{1}{r^2}(m+v_0 r),\ \frac{m}{r^2})} \tag{5.3}$$

To solve (5.1) for general *n*, we put $e^{-\eta} = t$, then the equation (5.1) reduces to

$$t\frac{d^2g}{dt^2} + (1 + \frac{m}{r^2}(t - \frac{v_0 r}{m} - 1))\frac{dg}{dt} - \frac{m}{r^2}png = 0 \tag{5.4}$$

We find the series solution of (5.4)

$$g(t) = \sum_{k=0}^{\infty} a_k\ t^{k+s},\ a_0 \neq 0 \tag{5.5}$$

together with boundary conditions:

$$\begin{aligned} g &= 1\ at\ t = 1 \\ g &= 0\ at\ t = 0 \end{aligned} \tag{5.6}$$

given by

$$g(t) = a_1\, t^{b_1 p}\ F\ (b_1 p\text{-}n;\ 1+b_1 p,\ \frac{-m}{r^2}pt) \tag{5.7}$$

where

$$a_1 = \frac{1}{F(b_1 p\text{-}n,\ 1 + b_1 p,\ \frac{-m}{r^2}p)}$$

$$b_1 = \frac{m}{r^2}(1 + \frac{v_0 r}{m})$$

and $F(a,b,x) = 1 + \frac{a}{b}x + \frac{a(+1)}{b(b+1)}\frac{x^2}{2} + \ldots$, the confluent hypergeometric series.

Special Case: When $p = \frac{n}{b_1}$, we have

$$F(b_1 p-n,\ 1+b_1 n,\ \frac{-m}{r^2}\, pt) = 1 \text{ for all } t, \text{ and } a_1 = 1.$$

Then,

$$g(\eta) = e^{-b_1 p\eta},\ g(0) = 1,\ g'(\eta) = -b_1 p\, e^{-b_1 p\eta},\ g'(0) = -b_1 p,$$

The **Nusselt** number is

$$Nu(x) = -\left(\frac{g'(\eta)}{g(\eta)}\right)_{\eta=0} = 1/r(m+v_0 r)p \tag{5.8}$$

6. Temperature Profile for Small p .

From the equation (5.7), we have

$$g(\eta) = a_1 e^{-b_1 pn}\Big[1 + \frac{m}{r^2}\, np\ e^{-\eta} + p^2\Big\{\frac{m}{4r^2}\ e^{-2\eta}(n^2-n)$$

$$-\frac{m^2}{r^2}(1+\frac{v_0 r}{m})\, e^{-\eta}\,(n+1)\Big\} + o(p^3)\Big].$$

Differentiating with respect to η and putting $\eta = 0$, we get

$$g'(0) = -P\Big[b_1 + \frac{m}{r^2}\, n + p\Big\{-\frac{m^2}{r^4}\,(\frac{n^2+n}{2} + (1+\frac{v_0 r}{m})n + \frac{v_0 r}{m} + 1)\Big\}$$

$$+ o(p^2)\Big] \tag{6.2}$$

The Nusselt number is

$$Nu(x) = -\left(\frac{g'(\eta)}{g(\eta)}\right)_{\eta=0} \cdot r$$

$$= \frac{p}{r}\,[m(n+1) + p\Big\{\frac{-m^2}{2}\,(\frac{n^2+n}{2} + (1+\frac{v_0 r}{m})\, n + \frac{v_0 r}{m} + 1)\Big\}$$

$$+ o(P^2)] \tag{6.3}$$

7. Conclusion and Discussion.

Taking the values of $\nu_0 = 0$ and $K_0^* = 0$ in the equation (3.8), we get $r = \sqrt{m}$ which is identical to $\left(\frac{u_p}{\nu x}\right)^{1/2}$ where $u_p = \alpha x$ in [2], in dimensional form and then the expressions (3.9), (3.10) and (4.6) reduce to the expressions of velocty components and temperature distribution obtained by Carragher, [3]. Also, we get:

(a) for $p = 1$; $Nu(x) = \left(\frac{u_p}{\nu x}\right)^{1/2} (e-1)^{-1}$

(b) for $p << 1$; $Nu(x) = \left(\frac{u_p}{\nu x}\right)^{1/2} p$ and

(c) for $p >> 1$; $Nu(x) = (u_p\, p/\nu x)^{1/2}$,

which are the results (4.21), (4.25) and (4.28) respectively of Carragher, [3].

(2) From (5 3), (5.8) and (6.3)

(a) for $p = 1$, $n = 0$;

$$Nu(x) = \left(\frac{u_p}{\nu x}\right)^{1/2} (e-1)^{-1}$$

(b) for $p = n$;

$$Nu(x) = p\left(\frac{u_p}{\nu x}\right)^{1/2}, \text{ and}$$

(c) for $p << 1$;

$$Nu(x) = p\left(\frac{u_p}{\nu x}\right)^{1/2} \left[(n+1) + p\left(-\frac{n^2}{2} - \frac{3n}{2} - 1\right) + o\,(p^2)\right]$$

which are the results (4.85), (4.86) and (4.87), respectively, of Carragher [3].

When the plate is rigid then $v_0 = 0$. The root of the equation (3.8) is $r = \dfrac{1}{(1/m)^{1/2} - K_0^*}$ and the results **(4.11), (4.12)** and (4.13) become

$$\text{(a) for } p = 1,\ Nu(x) = \frac{\left(\dfrac{m^2}{1-K_0^* m}\right)^{m^2/(1-K_0^* m)} exp\{-m^2/(1-K_0^* m)\}}{\gamma\left(\dfrac{m^2}{1-K_0^* m}, \dfrac{m^2}{1-K_0^*}\right)} \left(\frac{m}{1-K_0^* m}\right)^{1/2}$$

$$\text{(b) for } p << 1,\ Nu(x) \doteq \left(\frac{m^2}{1/m - K_0^*}\right)^{1/2} p, \text{ and}$$

$$\text{(c) } p >> 1,\ Nu(x) = m\left(\frac{mp}{2\,(1-K_0^* m)}\right)^{1/2},$$

where $Nu(x)$ is well defined for $K_0^* m < 1$ and dependency of $Nu(x)$ on visco-elasticity K_0^* is clarified. In case with plate temperature $T = T_s + Ax^n$, the results (5.3), (5.8) and (6 3) for the Nusselt number reduce to:

(a) for $p = 1,\ n = 0$;

$$Nu(x) = \frac{[m^2/(1-K_0^*)]^{(m^2/1-K_0^*)} \cdot exp\left\{-\dfrac{m^2}{(1-K_0 m)}\right\}}{\gamma\left(m/(1-K_0^* m), m/(1-K_0^* m)\right)} \left[m/(1-K_0^* m)\right]^{1/2}$$

(b) for $p = n/v_1$;

$$Nu(x) = n\left(\frac{m}{1-K_0^* m}\right)^{1/2}, \text{ and}$$

$$\text{(c)}\quad Nu(x) = p\left(\frac{m}{1-K_0^* m}\right)^{1/2}[m(n+1) - p(m^2/r^2(n^2+n/2+n+1))$$

$$+\, o\,(p^2)]$$

$$n = 0,\ Nu(x) = p\left(\frac{m}{1-K_0^* m}\right)^{1/2}\left[m - \frac{m^3}{1-K_0^* m}\,p + o(\,p^2)\right]$$

$$n = 1,\ Nu(x) = p\left(\frac{m}{1-K_0^* m}\right)^{1/2}\left[2m + \frac{m^3}{1-K_0^* m}\,p(-3) + o(p^2)\right]$$

$$n = 2,\ Nu(x) = p\left(\frac{m}{1-K_0^* m}\right)^{1/2}\left[3m - \frac{8m^3}{1-K_0^* m}\,p + o \cdot p^2)\right]$$

where the coefficient of convective heat transfer depends on n as well visco elasticity K_0^*.

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MEIJER'S G-FUNCTION AND FOX'S H-FUNCTION AND HEAT CONDUCTION IN A ROD UNDER TYPICAL BOUNDARY CONDITIONS

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ABSTRACT

In this paper, we employ Fox's H-function to derive a solution of a problem of heat conduction in a rod under typical boundary conditions.

1. Introduction.

The object of this paper is to employ an integral involving a Fox's H-function to obtain a solution of a problem of heat flow in a one-dimensional rod with constant thermal properties and no sources, when one end is held at zero temperature and the other end exchangs heat freely with the sourrounding medium. We derive a solution of the problem involving Meijer's G-function as a special case of our solution. We also obtain an anharmonic Fourier series for Fox's H-function.

The cases in which heat is produced in solids is becoming increasingly important in technical applications [1, pp. 11-12]. There are several applications of special functions in science and technology. Space research and nuclear reactor also give rise to several problems of the application of special functions. The applications of the special functions associated with the names of Legendre, Bessel, Hermite, Laguerre, Jacobi etc. in applied problems were given by several mathematicians from time to time.

Most of the practical problems, where elementary functions are used, are capable of being generalized with the help of generalized hypergeometric functions such as thegeneralized hypergeometric series, MacRobert's E-function, Meijer's G-function and Foxr's H-function. These generalizations may appear only mathematically interesting, and not appear to give meaningful physical interpretations at present. Nonetheless, it is expected that extensive study of the generalized hypergeometric functions together with the development of computers in future will open new frontiers of their applications woich are hitherto unknown [6, pp. 178-179]. However, some particular cases of our results may lead to meaningful physical interpretations.

Fox's H-function [6, p. 10] is a generalization of Meijer's G-function, which itself is a generalization of all special functions appearing in physical sciences and engineering [3, pp. 434-444]. Therefore, the solution obtanined in ihe paper is of a general character and hence may encompass several cases of interest. Our solution is a master or key formula from which a large number of solutions may be obtained for MacRrobert's E-function, Hypergeometric functions, Bessel functions, Legendre functions, Whittaker functions, Orthogonal polynomials and other related higher transcenedental functions [2, pp. 215-222]. The resnlts so derived may be found of great value for computing different values of $f(x)$ for several special functions.

The following modified form of the integral [4, p. 38, (2.5)], which is a particular case of an integral recently given by Mishra [4, p. 36, (2.1)] is required in the derivation of the solution :

$$(1.1)\quad \int_0^{\pi} \left(\sin\frac{\pi x}{L} \right)^{w-1} \sin\lambda \frac{x\pi}{L} \left[H^{u,v}_{p,q} \left[z \left(\sin\frac{x\pi}{L} \right)^{2h} \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx$$

$$= \frac{L}{2^{w\ 1} \sqrt{(\pi)}} \sin\lambda \frac{\pi}{2} \; H^{u,v+2}_{p+2,q+2} \left[z \,\middle|\, \begin{matrix} (1-w,\ 2h),\ (a_p, e_p) \\ (b_q.f_q)\left(\dfrac{1-w\pm\lambda}{2}, h \right) \end{matrix} \right],$$

where h is a positive number and

$$\sum_{j=1}^{p} a_j \; \sum_{j=1}^{q} b_j \leqslant 0, \; \sum_{j=1}^{v} e_j - \sum_{j=v+1}^{p} e_j + \sum_{j=1}^{u} f_j - \sum_{j=u+1}^{q} e_j \equiv A > 0,$$

$$|\arg z| < \tfrac{1}{2} A\pi \,;\; Re \left[w + 2h \min_{1 \leqslant j \leqslant u} b_j / f_j \right] > 0.$$

2. Heat Conduction in a Rod with One at zero Temperature and the Other End Exchanges Heat Freely with the Surrounding Medium at Zero Temperature

Let us now consider a simple, but typical problem of the so called outer heat conduction in a rod with one end held at zero temperature and the other end exchanges heat freely with the surrounding medium at zero temperature. If the thermal cofficients are constant and there are no sources of thermal energy, then temperature in a one-dimensional rod $0 \leqslant x \leqslant L$ satisfies:

$$(2.1)\quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \; t \geqslant 0 \,.$$

In view of the statement of the problem, the solution of this partial differential equation must satisfy the following conditions :

$$(2.2)\quad u(0,t) = 0\,;$$

(2.3) $\frac{\partial u}{\partial x}(L,t) + hu(L,t) = 0;$

(2.4) $u(x,t)$ is finite when $t \to \infty$.

(2.5) $u(x,0) = f(x) = \left(\sin\frac{\pi x}{L}\right)^{w-1} H_{p,q}^{u,v}\left[z \left(\sin\frac{\pi x}{L}\right)^{2h} \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right].$

3. The Solution of the Problem

The solution of the problem to be obtained is

$$(3.1) \quad u(x,t) = \frac{2^{2-w}}{\sqrt{(\pi)}} \sum_{n=0}^{\infty} \frac{\sin\frac{\pi}{2}\lambda_n \, e^{-(\pi\lambda_n/L)^2 kt} \sin \lambda_n \frac{\pi x}{2}}{\left(1 - \frac{1}{2\pi\lambda_n} \sin 2\pi\lambda_n\right)}$$

$$. \, H_{p+2,q+2}^{u,v+2}\left[z \middle| \begin{matrix} (1-w, 2h), (a_p, e_q) \\ (b_q, f_q), \left(\frac{1-w \pm \lambda_n}{2}, h\right) \end{matrix} \right],$$

valid under the conditions of (1.1).

Derivation. The solution of the problem stated in section 2 is given as [5, p. 77, (4)]:

$$(3.2) \quad u(x,t) = \sum_{n=1}^{\infty} B_n \sin \lambda_n \frac{\pi x}{L} e^{-(\pi\lambda_n/L)^2 kt},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are given as the roots of the transcendental equation :

$$\tan \pi\lambda_n = \frac{\pi\lambda_n}{hL} .$$

If $t = 0$, then by virtue of (2.5) and (3.2), we have

$$(3.3)\quad \left(\sin\frac{\pi x}{L}\right)^{w-1} H^{u,v}_{p,q}\left[z\left(\sin\frac{\pi x}{L}\right)^{2h}\middle|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right] = \sum_{n=1}^{\infty} B_n \sin\lambda_n\frac{\pi x}{L}.$$

Mulitiplying both sides of (3.3) by $\sin\lambda_m\frac{\pi x}{L}$ and integrating with respect to x from 0 to L, we get

$$\int_0^L \left(\sin\frac{\pi x}{L}\right)^{w-1}\sin\lambda_m\frac{\pi x}{L}\; H^{u,v}_{p,q}\left[z\left(\sin\frac{\pi x}{L}\right)^{2h}\middle|\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]dx$$

$$=\sum_{n=1} B_n\int_0^L \sin\lambda_n\frac{\pi x}{L}\,\sin\lambda_m\frac{\pi x}{L}\,dx.$$

Now using (1.1) and the following orthogonality property [5,p.28]:

$$\int_0^L \sin\lambda_n\frac{\pi x}{L}\sin\lambda_m\frac{\pi x}{L}\,dx = \begin{cases}0, & m\neq n;\\ \frac{L}{2}\left(1-\frac{1}{2\lambda_n\pi}\sin 2\lambda_n\pi\right), & m=n;\end{cases}$$

we obtain

$$(3.4)\quad B_m = \frac{\sin\lambda_m\frac{\pi}{2}}{2^{w-2}\sqrt{(\pi)}\left(1-\frac{1}{2\lambda_m\pi}\sin 2\lambda_m\pi\right)}$$

$$\cdot\, H^{u,v+2}_{p+2,q+2}\left[z\middle|\begin{matrix}(1-w,\ 2h),(a_p,e_p)\\(b_q,f_q),\left(\frac{1-w\pm\lambda_m}{2},\ h\right)\end{matrix}\right].$$

With the help of (3.2) and (3.4), the solution (3.1) follows immediately.

4. Special Cases

On specializing the parameters of the H-functions in (2.5) and (3.1), we can obtain many special cases of our solution (3.1) in view of [6, pp. 18-19]. However, we present below only one interesting special case of our solution.

In (2.5) and (3.1) reducing the H-functions into the G-functions, we obtain

$$(4.1)\quad u(x,0) = f(x) = \left(\sin\frac{\pi x}{L}\right)^{w-1} G^{u,v}_{p,q}\left[z\left(\sin\frac{\pi x}{L}\right)^{2h} \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right],$$

and

$$(4.2)\quad u(x,t) = \frac{1}{\sqrt{(\pi h)}} \sum_{n=0}^{\infty} \frac{\sin\frac{\pi}{2}\lambda_n\, e^{-(\pi\lambda_n/L)^2 kt} \sin\lambda_n\frac{\pi x}{L}}{\left(1-\frac{1}{2\pi\lambda_n}\sin 2\pi\lambda_n\right)}$$

$$.\; G^{u,v+2h}_{p+2h,q+2h}\left[z \middle| \begin{matrix} \triangle(2h,1-w),\, a_p \\ b_q,\, \triangle\left(\frac{1-w-\lambda_n}{2}\right),\ \triangle\left(h,\ \frac{-w+\lambda_n}{2}\right) \end{matrix} \right].$$

where h is a positive integer and $2(u+v)>p+q$, $|\arg z| < (u+v-\frac{1}{2}p-\frac{1}{2}q)\pi$, $Re(w+2hb_j) > 0$ ($j=1,\dots,u$).

5. The Anharmonic Fourier Series For Fox's H-Function

If we put $t = 0$ in (3.1), we obtain the following anharmonic Fourier series :

$$(5.1)\quad \left(\sin\frac{\pi x}{L}\right)^{w-1} H^{u,v}_{p,q}\left[z\left(\sin\frac{\pi x}{L}\right)^{2h} \middle| \begin{matrix} (a_p,e_p) \\ (b_q,f_q) \end{matrix} \right]$$

$$= \frac{2^{2-w}}{\sqrt{(\pi)}} \sum_{n=0}^{\infty} \frac{\sin\lambda_n\frac{\pi}{2}\ \sin\lambda_n\frac{\pi x}{L}}{\left(1-\frac{1}{2\pi\lambda_n}\sin 2\pi\lambda_n\right)}$$

$$.\; H^{u,v+2}_{p+2,q+2}\left[z \middle| \begin{matrix} (1-w,2h),\ a_p,e_p) \\ (b_q,f_q),\ \left(\frac{1\ w\pm\lambda_n}{2},\ h\right) \end{matrix} \right.$$

where in addition to the conditions of (1.1) $\lambda_1,\lambda_2,\dots,\lambda_m,\dots$ are the roots of the transendental equation

$$\tan\pi\lambda_n = \frac{\pi\lambda_n}{hL}.$$

Note: The reults (3.1) and (5.1) present an example of the relationship between a boundary value problem and an anharmoaic Fourier series for Fox's H-function.

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STABILITY OF STRATIFIED FLUIDS IN POROUS MEDIA IN THE PRESENCE OF HORIZONTAL MAGNETIC FIELDS AND ROTATION

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ABSTRACT

We have studied the Rayleigh-Taylor instability of stratified fluid in porous medium in the presence of a horizontal magnetic field and uniform rotation. The effect of coriolis force on the stability is chosen along the direction of the magnetic field and transverse to that of the gravitational field Both type of perturbations are considered in studying the growth rate of the disturbance. It is found that the fluid hgving negative density gradient in the upward direction remains stable with or without magnetic field and rotation. However for a fluid having positive density gradient the stratification is unstable for long wavelength of the distubaance. The overstability takes place in an intermediate wave-length range when transverse perturbations are considered.

1. Introduction

The character of equilibrium of an inviscid, incompressible fluid having variable density in the vertical direction was investigated by Rayleigh [5] was demonstrated that the system is stable or unstable according as the density decreases every where or increase anywhere. An experimental demonstration of the development of the Rayleigh-Taylor is desribed by Taylor [9]. Chandrasekhar [2] introduced the viscosity into such problem. Reid [6] studied the effects of surface tension and viscosity on the stability of two superposed fluids. Bellman and Penningtion [1] further investigated in detail illustrating the combined effects of viscosity and surface tension. Gupta [3] again studied the stability of a horizontal layer of a perfectly conducting fluid with continuous density and viscosity stratification in the presence of a horizontal magnetic field. The effect of a vertical magnetic field on the development of Rayleigh-Taylor instability was considered by Hide [4]. Sharma and Sharma [8] have studied the effect of uniform rotation on the stability of two superposed fluids in porous medium. The usual viscous term in the equation of the fluid motion actually arsies from the boundary conditions between fluid and solid. The penetration of a fluid into a porous medium was carried out by the Saffman and Taylor [7] to study the effect of porisity of the medium on the stability of fluids.

In the present paper we consider the Rayleigh-Taylor instability of a fluid with continuously density stratification in porous medium in the presence of a horizontal magnetic field. The fluid is considered to be inviscid, incompressible, infinitely coducting and finitely extending so that the free surface is almost horizontal. The fluid is acted on by gravity g $(0,0,-g)$, a uniform rotation Ω $(\Omega,0,0)$ and a horizontal magnetic field $\mathbf{H}$ $(H,0,0)$. The character of the equilibrium of this initial static state can be determined by disturbing the system slightly. Let $\delta\rho$, δp, $\mathbf{u}$ (u,v,w) and $\mathbf{h}$ $(h_x,h_y,h_z$ denote the perturbations in density presurre p, velocity $(0,0,0)$ and horizontal magnetic field. k_1 is the permeability of the medium according to Darcy's law. In the present paper our purpose is to deal with the stability of stratified fluid in porous medium in the presence of a magnetic field and uniform rotation along x-axis. The coriolis forces also play an important role on the stability of astrophysical problems and due to rotation taking along x-axis, both types of perturbations exist.

Then the linearized hydromagnetic perturbation equations are

$$\rho \frac{\partial \bar{u}}{\partial t} = - \bar{V}\delta p + \bar{g}\,\delta\rho - \frac{\mu}{k_1}\,\bar{u} + 2\,(\,\hat{u} \times \Omega\,) +$$

$$+ \frac{H}{4\pi}\,[\,(\bar{V} \times \bar{h}\,) \times \bar{\mathrm{H}}\,], \tag{1}$$

$$\bar{V} \cdot \bar{u} = 0, \tag{2}$$

$$\bar{V} \cdot \bar{h} = 0, \tag{3}$$

$$\frac{\partial \bar{h}}{\partial t} = \bar{V} \times (\,\bar{u} \times \bar{\mathrm{H}}), \tag{4}$$

$$\frac{\partial \delta\rho}{\partial t} + (\,\bar{u} \cdot \bar{V}\,)\,\rho = 0, \tag{5}$$

where k_1 is the permeability of the medium and μ the viscosity of the fluid and Ω is rotation along x-axis.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y, z and time t is given by

$$f(z)\,exp\,(ik_x x + ik_y y + nt), \tag{6}$$

where k_x, k_y are the horizontal wave numbers and n is the growth rate of the disturbance.

Equations (1) to (5), using expression (6), gives

$$\rho n u = - ik_x \delta p - \mu \frac{u}{k_1}, \tag{7}$$

$$\rho\, nv = -\, ik_v \delta p - \mu \frac{v}{k_1} + \frac{H}{4\pi} (ik_x h_v - ik_v h_x) + 2\rho\Omega w, \tag{8}$$

$$\rho n w = -\, D\delta p - \mu \frac{w}{k_1} + \frac{\mathrm{H}}{4\pi} (ik_x h_z - Dh_x) - g\delta\rho - 2\rho\Omega v, \tag{9}$$

$$ik_x u + ik_y\, y + Dw = 0, \tag{10}$$

$$ik_x h_x + ik_y h_y + Dh_z = 0, \tag{11}$$

$$n\delta\rho = w(D\rho) = 0, \tag{12}$$

$$nh_x = ik_x\, Hu, \tag{13}$$

$$nh_y = ik_x\, Hv, \tag{14}$$

$$nh_z = ik_x\, Hw, \tag{15}$$

where D stands for d/dz.

Eliminating δp between above equations. we get

$$(n^2 + \frac{nv}{k_1} + K_x^2 V^2)\, D^2 w + D\rho\, Dwn^2 - [k^2(n^2 + \frac{nv}{k_1} + k_x^2) + \\ + \frac{4\rho\, n^2\, \Omega^8}{n^2 + \frac{nv}{k_1} + k_x^2 V^2}\, k_x^2 - gk^2 D\rho]\, w + 2i\Omega\, nk_v\, D\rho\, w, \tag{16}$$

where $V^2 = H^2/4\pi\rho$, is the square of the Alfven velocity.
Here, $\mu/\rho = v$, denotes the kinematic viscosity of the fluid.

Equation (16) is the general equation for the discussion to consider the stability of stratified fluid in a porous medium.

2. The case of Exponentially varying density. $(k_y = 0)$

Let us assume the stratification in density of the form

$$\rho = \rho_1 e^{\beta z}\,, \tag{17}$$

where ρ_1 and β are constants.

Combining the equations (16) and (17), we get

$$(n^2 + n\frac{v}{k_1} + k_x^2 V^2,\, D^2 w + n^2 \beta Dw - [n^2 + n\frac{v}{k_1} + k_x^2 V^2)\, k^2 + \\ + \frac{4\Omega^2 n^2}{n^2 + n\frac{v}{k_1} + k_x^2 V^2}\, k_x^2 - g\beta k^2] = 0. \tag{18}$$

The solution of equation (18) is given by

$$w = A_1 e^{q_1 z} + A_2 e^{q_2 z} , \tag{19}$$

where A_1, A_2 are two arbitrary constants and q_1, q_2 are the roots of the equation

$$\left(n^2 + n\frac{\nu}{k_1} + k_x^2 V^2\right) q^2 + n^2 \beta q - \Bigg[\left(n^2 + n\frac{\nu}{k_1} + k_x^2 V^2\right) k^2 + \frac{4\Omega^2 n^2 k_x^2}{n^2 + n\frac{\nu}{k_1} + k_x^2 V^2} - g\beta k^2 \Bigg] = 0 . \tag{20}$$

If the fluid is supposed to be confined between two rigid planes at $z = 0$ and $z = d$, which implies that

$$(q_1 - q_2) d = 2im\pi . \tag{21}$$

Equation (20) yields the following dispersion relation :

$$n^4 [\beta^2 + k^2 + \frac{m^2\pi^2}{d^2}] + 8n^3 \frac{\nu}{k_1} [k^2 + \frac{m^2\pi^2}{d^2}] + 4n^2 [(\frac{\nu^2}{k_1^2} + 2k_x^2 V^2) (k^2 + \frac{m^2\pi^2}{d^2}) + 4\Omega^2 k_x^2 - g\beta k^2] + 4n\frac{\nu}{k_1} [2k_x^2 V^2 (k^2 + \frac{m^2\pi^2}{d^2}) - g\beta k^2] + 4k_x^4 V^4 (k^2 + \frac{m^2\pi^2}{d^2}) - g\beta k_x^2 V^2 = 0 \tag{22}$$

Equation (22) is biquadratic in n, therfore, it must give four roots. If $\beta < 0$, all the terms of equation (22) are positive, consequently, either n is real and negative and the system is always stable for all wavenumbers. If $\beta > 0$, the constant trem in equation (22) is negative, if

$$k^2 < \frac{g\beta \sec^2\phi}{V^2} - \frac{m^2\pi^2}{d^2} . \tag{23}$$

Equation (22), has at least one positive real root and so the system is unstable for all wavenumbers satisfying

$$k^2 < \frac{g\beta sec^2\phi}{V^2} - \frac{\pi^2 m^2}{d^2}$$, where ϕ is the angle between k_x and k ($k_x = k\, cos\phi$) .

Case II Transerverse perturbation ($k_x = 0$)

When $k_x = 0$, we find that the equation (16) reduces to

$$(n^2 + \frac{n\nu}{k_1})\, D^2 w + DwD\rho\, n^2 - [k^2 n^2 + \frac{n\nu}{k_1}) + \frac{4n^2\Omega^2}{n^2 + \frac{n\nu}{k_1}}$$

$$- gk^2 D\rho]\, w + 2i\Omega n k_y D\rho\, w = 0. \tag{24}$$

In this section, we shall consider the case of a continuously stratified fluid of depth d in which density distribution is given by

$$\rho = \rho_0\, exp\, \beta z, \tag{25}$$

where ρ_0 and β are constants.

If the fluid is supposed to be confined between two rigid planes at $z = 0$ and $z = d$, equation (24) reduces to

$$n^3[\beta^2 + 4k^2 + \frac{4m^2\pi^2}{d^2}] + 8n^2[\frac{\nu}{k_1}(\frac{m^2\pi^2}{d^2} + k^2) - i\Omega k\beta] + 4n[\frac{\nu^2}{k_1^2}$$

$$(\frac{m^2\pi^2}{d^2} + k^2) - \frac{2i\Omega\beta k}{k_1} - g\beta k^2] - 4g\beta k^2 \frac{\nu}{k_1} = 0 \tag{26}$$

Special Case : $\frac{1}{k_1} = 0.$

When the effects of permeability is not considered then the equation (26) reduces to

$$n^2[\beta^2 + 4k^2 + \frac{4m^2\pi^2}{d^2}] - 8nik\beta\Omega - 4g\beta k^2 + 0. \tag{27}$$

The equation (27) can be written as

$$n = \frac{i\beta k\, \Omega}{\alpha^2 + k^2} \pm ik\, [\frac{\Omega^2 \beta^2}{(\alpha^2 + k^2)^2} - \frac{g\beta}{\alpha^2 + k^2}]^{1/2}. \tag{28}$$

For the stability of the configuration, the quantity under the root sign in equation (28) must be positive. For this to occur, k, the wave number of the disturbance, must be less than the critical value k given by

$$-k_*^2 = \frac{\beta\Omega^2}{g} - \alpha^2. \tag{29}$$

Overstability takes place for $k > k_*$. It can be seen here that rotation stabilizes small wave numbers or large wavelengths.

Numerical calculations are performed to locate the roots of n from equation (28) to ascertain the influence of rotation on the growth rate of the unstable configuration. These calculations are performed for different values of rotation. In order to ebtain the positive value of n from equation (28) we first reduces to the nondimensional form by measuring n and Ω in terms of $\sqrt{g}$. For $\alpha = 1$, these calculation are presented in Table I, where we have given the growth rate (positive value of n) against wave number k, from $k = 0$ to $k = 5.0$

It can be seen from the table that the overstability takes place in an intermediate wavelength range It is further observed that the overstable range $0 < k < k_*$ is wiped out due to inclusion of strong rotation and is rendered stable. However, again another over-stable $k_1 < k < k_2$ for large wave numbers is manifested.

Table 1. The following table represents the value of n (growth rate) against k, for $\alpha = 1$ and $\beta = 1$ and different values of rotation.

S.N.	k	$\Omega = 1$	$\Omega = 3$
		n	n
1	0.1	$.0000 + 0.3691i$	$.0000 + 0.5665i$
2	0.2	$.0000 + 0.7144i$	$.0000 + 1.1214i$
3	0.5	$.0000 + 1.4893i$	$.0000 + 2.3335i$
4	0.8	$.0000 + 1.8849i$	$.0000 + 2.7824i$
5	1.0	$.0000 + 1.7671i$	$.0000 + 2.8220i$
6	1.2	$.0000 + 1.6820i$	$.0000 + 3.1617i$
7	1.5	$.0000 + 1.2708i$	$.0000 + 2.4898i$
8	1.7	$.0000 + 1.0164i$	$.0000 + 2.3246i$
9	1.8	$.2002 + 0.8488i$	$.0000 + 2.1990i$
10	2.0	$.4000 + 0.8000i$	$.0000 + 2.0000i$
11	2.2	$.5618 + 0.7532i$	$.0000 + 1.7989i$
12	2.5	$.6270 + 0.6895i$	$.0000 + 1.4897i$
13	2.8	$.6969 + 0.6333i$	$.1527 + 0.9500i$
14	3.0	$.7347 + 0.i6000$	$.3000 + 0.9000i$
15	4.0	$.8489 + 0.4684i$	$.6656 + 0.7056i$
16	5.0	$.9100 + 0.3846i$	$.7932 + 0.5234i$

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APPLICATION OF THE MULTIVARIABLE H-FUNCTION IN HEAT CONDUCTION IN A ROD UNDER THE ROBIN CONDITION

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ABSTRACT

The aim of this paper is to make use of the mulivariable H-function in obtaining a solution of the partial differential equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

related to a problem of heat condnction in a rod under the Robin condition.

1. INTRODUCTION

As an example of the multivariable H-function in applied mathematics, we shall consider the problem of obtaining a solution of a problem of flow of heat in a uaiform rod under third kind of boudary conditions. Consider the problem of heat coduction in a uniform rod under the Robin condition (at zero temperature, radaition at the ends inside the medium). If the the thermal coefficient is constant and no source of thermal energy then a function $u(x,t)$ representing the tamperature in a rod $0 \leqslant x \leqslant L$ must satisfy the heat equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} \qquad \ldots (1)$$

and the initial conditons

$$u(x,0) = f(x) \qquad \ldots (2)$$

$$\frac{\partial \bar{u}}{\partial t}(0,t) - h\, u(0,t) = 0 \qquad \ldots (3)$$

$$\frac{\partial u}{\partial x}(L,t) + h\,u(L,t) = 0;\ h > 0 \qquad \dots (4)$$

Here we shall consider

$$f(x) = (sin\frac{\pi x}{L})^{w-1}\ H^{0,\lambda:\,(u',v');\dots;(u^{(r)},v^{(r)})}_{A,C:\,[B',D'];\dots;[B^{(r)},D^{(r)}]}\begin{pmatrix} z_1 & (sin\ \frac{\pi x}{L})^{2k_1} \\ \vdots & \\ z_r & (sin\ \frac{\pi x}{L})^{2k_r}\end{pmatrix} \dots (5)$$

where the mulitivariable H-function is the H-function of several complex variables introduced by Srivastava and panda ([3]; see also [4]; p. 251).

The expression $(\ p\ cos\ E_n x + q\ sin\ E_n x\ )exp(\ -\ \mu E_n^2\ t)$ ...(6)

satisfies the equation (1). The equation (6) also satisfies (3) and (4) provided that $E_n q - hp = 0$ ...(7)

and $E_n(q\ cos\ E_n L - p\ sin\ E_n L) + h(\ p\ cos\ E_n L + q\ sin\ E_n L) = 0$ ... (8)

From (6) and (7), we get $\dfrac{A}{B} = \dfrac{E_n}{h}$ ... (9)

and $tan\ E_n L = 2E_n h/(E_n^2 - h^2)$, ... (10)

where E_n is the n^{th} positive root of (14).

Then, the solution of our problem takes the following elegant form

$$u(x,t) = \sum_{n=1}^{\infty} R_n\,(cos\ E_n x + \frac{h}{E_n}\ sin\ E_n x)exp(\ \mu E_n^2\ t) \qquad \dots (11)$$

The following results will be required:

$$\int_0^L (\ sin\ \frac{\pi x}{L})^{w-1}\ sin\frac{E_n\pi x}{L}\ dx = L\ 2^{-w\;\,1}\ sin\ \frac{E_m\pi}{2}\ \Gamma(w)$$

$$\left\{\frac{(w \pm E_m + 1)}{2}\right\}^{-1},\ Re(w) > 0. \qquad \dots (12)$$

$$\int_0^L (\ cos\ \frac{E_m\pi x}{L}\ (\ sin\ \frac{\pi x}{L})^{w-1}\ dx = L\ 2^{-w+1}\ cos\frac{E_m\pi}{2}\ \Gamma(w)$$

$$\left\{\frac{(w \pm E_m + 1)}{2}\right\}^{-1},\ Re(w) > 0. \qquad \dots (13)$$

$$\int_0^L \left(\cos E_n x + \frac{h}{E_n} \sin E_n x \right) \left(\cos E_m x + \frac{h}{E_m} \sin E_m x\right) dx$$

$$= \left\{ \begin{array}{ll} (2E_n^2)^{-1} \left[(E_n^2 + h^2) L + 2h\right] , & m = n \\ 0 & , m \neq n \end{array} \right\} \qquad \dots (14)$$

where E_n is positive root of the equation $\tan EL = 2hE/(E^2 - h^2)$. ... (15)

2. The Main Integral Formulae

$$\int_0^L \left(\sin \frac{\pi x}{L} \right)^{w-1} \sin\frac{E_m \pi x}{L}\, H^{0,\lambda:\,(u',v');\dots;(u^{(r)},v^{(r)})}_{A,C:\,[B',D'];\dots;[B^{(r)},D^{(r)}]} \left(\begin{array}{c} z_1 \left(\sin\frac{\pi x}{L}\right)^{2k_1} \\ \vdots \\ z_r \left(\sin\frac{\pi x}{L}\right)^{2k_r} \end{array} \right) dx$$

$$= L\, 2^{1-w} \sin \frac{E_m \pi}{2}\, H^{0,\lambda+1\,:\,(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1,C+2:\,[B',D'];\dots;[B^{(r)},D^{(r)}]} \left(\begin{array}{l} [1-w:2k_1, \quad ,2k_r], \\ \left[(c):\psi', \dots, \psi^{(r)}\right], \end{array} \right.$$

$$\left. \begin{array}{ll} [(a):\theta',\dots,\theta^{(r)}]:\ [(b'):\phi'];\dots;[b^{(r)}):\phi^{(r)}]\ ; & \begin{array}{c} z_1\ 4^{k_1} \\ \vdots \end{array} \\ \left[\frac{1\ w \pm E_m}{2}:k_1,\dots,k_r\right]:[(d')\cdot\delta');\dots,[\,d^{(r)}):\delta^{(r)}] & z_r\ 4^{k_r} \end{array} \right), \qquad \dots (16)$$

where $k_i > 0$, $Re\left(w + \sum_{i=1}^{r} k_i\, d_j^{(i)} / \delta_j^{(i)}\right) > 0$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant u^{(i)}$,

$|\arg (z_i)| < T_i \pi/2$, $T_i > 0$.

$$\int_0^L \left(\sin \frac{\pi x}{L} \right)^{w-1} \cos\frac{E_m \pi x}{L}\, H^{0,\lambda:(u',v');\dots;(u^{(r)},v^{(r)})}_{A,C:[B'\ D\ ];\dots;[B^{(r)},D^{(r)}]} \left(\begin{array}{c} z_1 \left(\sin\frac{\pi x}{L}\right)^{2k_1} \\ \vdots \\ z_r \left(\sin\frac{\pi x}{L}\right)^{2k_r} \end{array} \right) dx$$

$$= L\, 2^{1-w} \cos \frac{E_m \pi}{2}\, H^{0,\lambda+1\,:\,(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D\ ];\dots;[B^{(r)},D^{(r)}]} \left(\begin{array}{l} [1-w:2k_1,\dots,2k_r]\ ; \\ \left[(c):\psi',\dots,\psi^{(r)}\right] \ . \end{array} \right.$$

$$\left. \begin{array}{ll} [(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi'];\dots;[B^{(r)}):\phi^{(r)}]; & \begin{array}{c} z_1\ 4^{k_1} \\ \vdots \end{array} \\ \left[\ \frac{1-w \pm E_m}{2}:k_1,\dots,k_r\right):[(d\ ):\delta'] \ ;\dots;\ [(d^{(r)}):\delta^{(r)}]; & z_r\ 4^{k_r} \end{array} \right), \qquad \dots (17)$$

where $k_i > 0$, $Re\left(w + \sum_{i=1}^{r} k_i\, d_j^{(i)} / \delta_j^{(i)}\right) > 0$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant u^{(i)}$,

$|\arg (z_i)| < T_i \pi/2$, $T_i > 0$.

The integral formulae in (16) and (17) can be established easily by mnking use of the definition of the multivariable H-function ([3], see also [4]) and the result in (12) and (13) respactively.

3. Solution of the Problem

The solution of the problem to be obtained is

$$u(x,t) = \frac{2L}{2^{w-1}} \sum_{n=1}^{\infty} E_n^2 \; exp\;(-\mu E_n^2\; t).(cos E_n x + \frac{h}{E_n}\; sin\; E_n x)$$

$$\left\{ [(E_n^2 + h^2)L + 2h \right\}^{-1}$$

$$\cdot \left\{ cos\;\frac{E_n\pi}{2}\; H^{0,\lambda+1\;:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{array}{l} [1-w:2k_1,\ldots,2k_r], \\ [(c):\psi',\ldots,\psi^{(r)}], \end{array} \right. \right.$$

$$\left. \begin{array}{ll} [(a):\theta',\ldots,\theta^{(r)}]:[b'):\phi';\ldots;[b^{(r)}):\phi^{(r)}]; & z_1\,4^{k_1} \\ & \vdots \\ [\frac{1\;w\pm E_n}{2}:k_1,\ldots,k_r]:[(d'):\delta'];\ldots;[(d^{(r)})\delta^{(r)}]; & z_r\,4^{k_r} \end{array} \right)$$

$$+ \frac{h}{E_n} sin \frac{E_n\pi}{2}\; H^{0,\lambda+1\;:(u',v');\ldots;(u^{(r)})}_{A+1,C+2;[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{array}{l} [1-w:2k_1,\ldots,2k_r], \\ [(c):\psi',\ldots,\psi^{(r)}], \end{array} \right.$$

$$\left. \left. \begin{array}{ll} [(a)\cdot\theta',\ldots,\theta^{(r)}],[(b'):\phi'];\ldots;[b^{(r)}):\phi^{(r)}]; & z_1\,4^{k_1} \\ & \vdots \\ [\frac{1-w\pm E_n}{2}:k_1,\ldots,k_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; & z_r\,4^{k_r} \end{array} \right) \right\}, \quad \ldots (18)$$

where $k_i > 0$, $Re(w + \sum_{i=1}^{r} k_i\, d_j^{(i)} / \delta_j^{(i)}) > 0$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant u^{(i)}$.

$|arg\,(z_i)| < T_i\pi/2$, $T_i > 0$.

4. Derivation of (18)

If $t=0$, then from (5) and (11), we have

$$(sin\,\frac{\pi x}{L})^{w-1}\;\; H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{array}{l} z_1\;(sin\,\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r\;(sin\,\frac{\pi x}{L})^{2k_r} \end{array} \right)$$

$$= \sum_{n=1}^{\infty} R_n\,(cos\,E_n x + \frac{h}{E_n}\; sin\,E_n x) \qquad \ldots (19)$$

Multiplying both sides of (19) by $(\cos E_m x + \frac{h}{E_m} \sin E_m x)$, integrate with respect to x from 0 to L and use (14), (16) and (17), we thus obtain

$$R_m = 2L\, E_m^2\, 2^{1-w} \left\{ [(E_m^2 + h^2)L + 2h] \right\}^{-1}$$

$$\Bigg\{ \cos\frac{E_m\pi}{2}\, H^{0,\lambda+1:\,(u',v');\ldots;(u^{(r)},\,v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{array}{l|l} [1-w:2k_1, \;,2k_r],[(a'):\theta',\ldots,\theta^{(r)}]:\,[(b'):\phi'];\ldots;[b^{(r)}):\phi^{(r)}]\;; & z_1\, 4^{k_1} \\ [(c):\psi',\ldots,\psi^{(r)}],\,[\frac{1\;\,w\pm E_m}{2}:k_1,\ldots,k_r]:[(d'):\delta');\ldots,[(d^{(r)}):\delta^{(r)}] & \vdots \\ & z_r\, 4^{k_r} \end{array} \right)$$

$$+ \frac{h}{E_m} \sin\frac{E_m\pi}{2}\, H^{0,\lambda+1:\,(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{array}{l} [1-w:2k_1,\ldots.2k_r], \\ [(c):\psi',\ldots,\psi^{(r)}] \;, \end{array} \right.$$

$$\left. \begin{array}{l|l} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[B^{(r)}):\phi^{(r)}]; & z_1\, 4^{k_1} \\ [\frac{1-w\pm E_m}{2}:k_1,\ldots,k_r):[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; & \vdots \\ & z_r\, 4^{k_r} \end{array} \right) \Bigg\}, \quad \ldots (20)$$

Substituting the value of R_m from (20) in (19), we get the desired solution.

5. Special Cases

For $r=1$, the results in (16), (17) and (18) reduces to the following results involving Fox's H-function [2]

$$\int_0^L \left(\sin\frac{\pi x}{L} \right)^{w-1} \cos\frac{E_m x\pi}{L}\, H^{u,v}_{B,D}\left[z\left(\sin\frac{\pi x}{L} \right)^{2k} \left| \begin{array}{l} [(b):\phi] \\ [(d):\delta] \end{array} \right. \right] dx$$

$$= L\, 2^{1-w} \sin\frac{E_m\pi}{2}\, H^{u,v+1}_{B+1,D+2}\left[z\, 4^k \left| \begin{array}{l} [1-w:2k],\,[(b):\phi] \\ [(d):\delta],[\frac{1-w\pm E_m}{2}:k] \end{array} \right. \right], \quad \ldots (21)$$

where $k>0$, $Re(w+k\, d_j/\delta_j)>0$, $1 \leqslant j \leqslant u$, $|\arg z| < T\pi/2$.

$$\left(T = \sum_1^{v} \phi_i - \sum_{v+1}^{B} \phi_i + \sum_1^{u} \delta_i - \sum_{u+1}^{D} \delta_i \right)$$

$$\int_0^L \left(\sin \frac{\pi x}{L} \right)^{w-1} \cos \frac{E_m \pi x}{L} \; H^{u,v}_{B,D} \left[z \left(\sin \frac{\pi x}{L} \right)^{2k} \middle| \begin{matrix} [(b):\phi] \\ [(d):\delta] \end{matrix} \right] dx$$

$$= L\, 2^{1-w} \cos \frac{E_m \pi}{2} \; H^{u,v+1}_{B+1,D+2} \left[z\, 4^k \middle| \begin{matrix} [1-w:2k),\ [(b):\phi] \\ [(d):\delta],\ [\frac{1-w \pm E_m}{2} :k] \end{matrix} \right], \quad \ldots(22)$$

where $k > 0$, $Re(w + k\, d_j/\delta_j) > 0$, $1 \leqslant j \leqslant u$, $|\arg z| < T\pi/2$. $T > 0$.

$$u(x,t) = 2\, L 2^{1-w} \sum_{n=1}^{\infty} E_n^2 \; exp\left(-\mu E_n^2\, t \right). \left(\cos E_n x + \frac{h}{E_n} \sin E_n x \right)$$

$$\left\{ [(E_n^2 + h^2) L + 2h] \right\}^{-1}$$

$$\cdot \left\{ \cos \frac{E_n \pi}{2} \; H^{u,v+1}_{B+1,D+2} \left[z\, 4^k \middle| \begin{matrix} [1-w:2k],[(b):\phi) \\ [(d):\delta],\ [\frac{1-w \pm E_n}{2} : k] \end{matrix} \right] \right.$$

$$\left. + \frac{h}{E_n} \sin \frac{E_n \pi}{2} \; H^{u,v+1}_{B+1,D+2} \left[z\, 4^k \middle| \begin{matrix} [1-w:2k]\ [(b):\phi] \\ [(d):\delta].\ [\frac{1-w \pm E_n}{2} : k] \end{matrix} \right] \right\}, \quad \ldots \quad (23)$$

where $k > 0$, $Re(w + k\, d_j/\delta_j) > 0$. $1 \leqslant j \leqslant u$ $|\arg z| < T\pi/2$, $T > 0$.

6. Concluding Remark

In view of the generality of the multivariable H-function involved, the results derived in this paper are of a general character and may prove to be useful in several interesting situations appearing in the literature on applied mathematics and mathematical physics.

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IDEAL IN SEMIPRIME NONASSOCIATIVE RINGS

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ABSTRACT

The following theorem is proved: Let R be a semiprime non-associative ring. If A is a right (or left) indeal of R contained in the nucleus and the associator ideal of R, then $A = (0)$.

1. INTRODUCTION

A non-associative ring does not presuppose the associative law of multiplication. The nucleus N of an arbitrary non-associative ring R consists of all elements x in R such that

$$(x,y,z) = 0 = (y,z,x) = (z,x,y) \quad \text{for all } y,z \text{ in } R$$

where the associator (x,y,z) is defined by

$$(x,y,z) = (xy)\,z - x\,(yz).$$

It is well known (see Schafer [3], p. 13) than N is an associative subring of R. All rings R have an ideal, called the associator ideal. It is defined as the smallest ideal which contains all associators, It actually consists of all finite sums of associators and right or left multiples of associators (see Kleinfeld [1], [2]). The associator ideal is never zero, except when R is associative. A ring R is called semiprime if R has no nonzero ideal squaring to zero. A straightforward verification shows that any ring satisfies

$$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z,$$

which is known as the Teichüller identity.

2. THE MAIN RESULT

we prove the following:

Theorem. Let R be a semiprime nonassociative ring. If A is a right (or left) idel of R contained in the nucleus and the associator ideal of R, then $A = (0)$.

Proof. We shall use the symbol R (or A) to represent an element of R (or A). When this notation is used, we mean to consider not only all possible expressions where R's (or A's) are replaced by elements of R (or A), but the additive subgroup generated by these. Let $I = (R,R,R) + (R,R,R)R$ be the associator ineal of R. Now $A \subseteq N$ implies that $(A,R,R) = (R,A,R) = (R,R,A) = (0)$. For any a in A and x,y,z in R, by the Teichmüller identity,

$(ax,y,z) - (a,xy,z) + (a,x,yz) - a(x,y,z) - (a,x,y)z = 0$. If A is a right ideal, (ax,y,z), (a,xy,z), (a,x,yz), $(a,x,y) \in (A,R,R) = (0)$. Therefore, $a(x,y,x) = 0$. Since $A \subseteq N$, we have $A((R,R,R)R) = (A(R,R,R)R) = (0)$. This implies that $A((R,R,R)+(R,R,R)R) = (0)$. Thus, $AI = (0)$. Now $R(A + RA) \subseteq RA + R(RA) \subseteq RA + (RR)A - (R,R,A) \subseteq A+RA$. Also $(A+RA)R \subseteq AR + (RA)R \subseteq A + R(AR) \subseteq A+RA$. Thus if $<A>$ is the ideal generated by A then $A+RA = <A>$. Therefore, $<A>\ I = (A+RA)I \subseteq AI + (RA)I \subseteq R(AI) \subseteq (0)$. Moreover $A \subseteq I$ and I is an ideal implies that $<A> \subseteq I$. Thus $<A><A> = 0$, New let $I = (R,R,R) + R(R,R,R)$ be the associator ideal of R. Again for any a in A and w,x,y in R, by the Teicnmüller identity, $(wx,y,a) - (w,xy,a)$. $(w,x,y,a) - w(x,y,a) - (w,x,y)a = 0$. If A is a left ideal, $(wx\,y,a)$, (w,xy,a), (w,x,ya), $w(x,y,a) \in (R,R,A) = (0)$. Therefore, $(w,x,y)a = (0)$. Since $A \subsetneq N$, we have $(R(R,R,R)A = R((R,R,R)A) = (0)$. This implies that $((R,R,R)R + R(R,R,R))A = (0)$. Thus

$((R,R,R)R + R(R,R,R))A = (0)$. Thus, $IA = (0)$. Now $(A + AR)R \subseteq AR + (AR)R \subseteq AR + A(RR) + (A,R,R) \subseteq A + AR$. Also $R(R + AR) \subseteq RA + R(AR) \subseteq A + (RA)R \subseteq A + AR$. Thus if $<A>$ is the ideal generated by A, then $A + AR = <A>$. Therefore, $I <A> = I(A + AR) \subseteq IA + I(AR) \subseteq (IA)R \subseteq (0)$. Moreover $A \subseteq I$ and I is an ideaal implies that $<A> \subseteq I$. Thus $<A> <A> = (0)$.

We have shown that if A is a right (or left) ideal of R then $<A> <A> = (0)$. But R is semi-prime and $<A>$ is an ideal of R. This implies that $<A> = (0)$. Hence $A = (0)$.

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ON MULTIDIMENSIONAL MELLIN CONVOLUTIONS AND THE H-FUNCTION TRANSFORM

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ABSTRACT

A relation involving the Mellin convolution of the Multidimensional *H*- function transforms of two functions for *r* variables and the mulitidimensional *H*-function transform of the Mellin convolution of these functions is obtained in this paper. Results involving the Mellin convolution of the multidimensional Laplace transforms are obtained as particular cases.

1. Introduction and Definitions

Following the technique of Srivastava and Buschman [3], who studied relations involving the *H*-function transform and the Mellin convolution of two functions of a single variable, we obtain a relation involving the Mellin convolution of the mulidimensional *H* function transforms of two functions of *r* variables and the multidimensional H-function transform of the Mellin convolution of these functions. Results involving the Mellin convolution of the mulitidimensional Laplace transform are obtained as particular cases. In what follows, we shall use **Z** to represent the vector $[z_1, z_2, \ldots, z_r]$. Thus, **X** stands for $[x_1, x_2, \ldots, x_r]$, **- S** for $[-s_1, \ldots, -s_r]$ etc.

(a) The Multivariable H-function of Srivastava and Panda [5] is defined and represented as follows (see [4,p.251,eqns.(C-1) to (C-3)]:

$$H[z_1,\ldots,z_r] = H[\mathbf{Z}] = H^{0,n:\, m_1,n_1;\ldots;m_r n_r}_{p,q:\, p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix}(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}: \\ (b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,p_1};\ldots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (d'_j,\delta'_j)_{1,q_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r}\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L1}\cdots\int_{Lr}\phi_1(\xi_1)\ldots\phi_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)\, z_1^{\xi_1}\ldots z_r^{\xi_r}\, d\xi_1\ldots d\xi_r \quad \ldots(1.1)$$

where $\omega = (-1)^{1/2}$.

For definitions of the ϕ and ψ functions involved in (1.1), and for the conditions of existence of the H-function $H[\mathbf{Z}]$, we refer to [4,p.251–253]. If we take $n = 0$ in (1.1), we write

$$H[\mathbf{Z}]\,|_{n=0} = H_1[\mathbf{Z}]. \quad \ldots (1.2)$$

(b) The multidimensional integral transforms:

We study the following multidimensional integral transforms of a function $f(\mathbf{X})$ in this paper. Each of these multimensionl integral transforms would follow as a special or limiting case of one or the other of the two classes of multidimensional integral transforms introduced and studied systematically by Srivastava and Panda [6].

(1) The multidimensional Mellin transform:

Let

$$M\{f(\mathbf{X});\mathbf{S}\}$$

$$= \int_0^\infty \cdots \int_0^\infty x_1^{s_1-1}\ldots x_r^{s_r-1} f(\mathbf{X})\, dx_1\ldots dx_r = F(\mathbf{S}), \quad \ldots (1.3)$$

denote the multidimensional Mellin transform of $f(\mathbf{X})$ [6], provided that the rth order integral in (1.3) is absolutely convergent.

Its inversion formula is given by [6]

$$f(\mathbf{X}) = \frac{1}{(2\pi\omega)^r}\int_{c_1-\omega\infty}^{c_1+\omega\infty}\cdots\int_{c_r-\omega\infty}^{c_r+\omega\infty} x_1^{-s_1}\ldots x_r^{-s_r} F(\mathbf{S})\, ds_1\ldots ds_r; \quad \ldots(1.4)$$

$$x_1,\ldots,x_r > 0.$$

We require the following convolution property of the multidimensional Mellin transform (which can be proved easily) :

If $F(\mathbf{S})$ and $G(\mathbf{S})$ denote the multidimensional Mellin transform of $f(\mathbf{X})$ and $g(\mathbf{X})$, then

$$M\ \{ f(\mathbf{X})_* \, g(\mathbf{X}) : \mathbf{S} \} = F(\mathbf{S})\ G(\mathbf{S}), \qquad \ldots (1.5)$$

where

$$f(\mathbf{X})_* \, g(\mathbf{X}) = \int_0^\infty \cdots \int_0^\infty f\left(\frac{x_1}{u_1}, \ldots, \frac{x_r}{u_r}\right) g(u_1, \ldots, u_r)\ \frac{du_1}{u_1} \cdots \frac{du_r}{u_r} \qquad \ldots (1.6)$$

(ii) The multidimensional H-function transform :

The multidimensional H-function transform of $f(\mathbf{X})$ studied here is defined and represented as follows (and differs slightly from that used by Srivastava and Panda [6]) :

$$\hat{f}(\mathbf{U}) = \int_0^\infty \cdots \int_0^\infty x_1^{-1} \cdots x_r^{-1}\ H_B^A \left[\begin{array}{c|c} u_1 x_1 & C \\ \vdots & \\ u_r x_r & D \end{array} \right] f(\mathbf{X})\ dx_1 \ldots dx_r$$

$$= H_{B,}^{A'} {}_{\{D\}}^{\{C\}}\ [\, f(\mathbf{X}\ ;\ \mathbf{U}\,] \qquad \ldots (1.7)$$

where $\mathbf{U} = [u_1, \cdots, u_r]$ and

$A \equiv 0,0 : m_1, n_1 ; \ldots ; m_r\ n_r$

$A \equiv p,q : p_1, q_1 ; \cdots ; p_r, q_r$

$C \equiv (a_j ; \alpha'_j, \ldots, \alpha_j^{(r)})_{1,p} : (c'_j\ \gamma'_j)_{1,p1} ; \ldots ; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r}$

$D \equiv (b_j ; \beta'_j, \ldots, \beta_j^{(r)})_{1\,q} : (d'_j, \delta'_j)_{1,\,q_1} ; \ldots ; (d_j^{(r)}, \delta_j^{(r)})_{1,\,q_r}$,

and provided that the rth order integral in (1.7) is absolutely convergent, (iii) The multidimensional Laplace Transform:

$$L\{ f(\mathbf{X}) ; \mathbf{U}\}$$

$$= \int_0^\infty \cdots \int_0^\infty exp\left(-\sum_{i=1}^{r} u_i x_i\right) f(\mathbf{X})\ dx_1 \ldots dx_r = f(\mathbf{U}), \qquad \ldots (1.8)$$

denote the multimensional Laplace transform of $f(\mathbf{X})$ [2].

We have the following relation :

$$H_{0,0\,:\,0,1\,;\ldots;\,0,1,\ \{-\,:\,(\varepsilon_1,1)\,;\ldots;\,(\varepsilon_r,1)\}}^{0,0\,:\,1,0\,;\ldots;\,1,0,\ \{-\,:\,-\,;\ldots;\,-\}}\ [f(\mathbf{X})\,;\,\mathbf{U}]$$

$$= u_1^{\varepsilon_1}\ldots u_r^{\varepsilon_r}\ L\,\{x_1^{\varepsilon_1-1}\ldots x_r^{\varepsilon_r-1}\,f(\mathbf{X})\,;\,\mathbf{U}\}\,. \qquad \ldots\ (1.9)$$

(iv) The mulidimensional Meijer–Bessel transform :

Let

$$K\,\{f(\mathbf{X})\,;\,\nu_1,\ldots,\nu_r\,;\,\mathbf{U}\}$$

$$= \int_0^\infty\ldots\int_0^\infty \prod_{i=1}^{r}\,[(u_i x_i)^{1/2}\,K_{\nu_i}\,(u_i x_i)]\ f(\mathbf{X})\ dx_1\ldots dx_r \qquad \ldots\ (1.10)$$

denote the multidimensional Meijer–Bessel transform of $f(\mathbf{X})$, where $K_\nu(x)$ is the Bessel function of the second kind. If we put

$\nu_i = \pm\,1/2\,,\ i = 1,\ldots,r$ in (1.10), we have

$$K\,\{f(\mathbf{X})\,;\,\pm\,1/2\,,\ldots,\,\pm\,1/2\,;\,\mathbf{U}\} = (\pi/2)^r\,L\,\{f(\mathbf{X})\,;\,\mathbf{U}\}. \qquad \ldots(1.11)$$

We, also have

$$H_{0,0\,:\,0,2\,;\ldots;\,0,2\ \{-\,:\,(\mu_1,1),\,(\nu_1,1)\,;\ldots;\,(\mu_r,1),\,(\nu_r,1)\}}^{0,0\,:\,2,0\,;\ldots;\,2,0\ \{-\,:\,-\,-\,;\ldots;\qquad -\qquad -\}}\ [f(\mathbf{X})\,;\,\mathbf{U}\}]$$

$$= 2^{3r/2}\,\prod_{i=1}^{r}\,\{u_i^{\left(\frac{\mu_i+\nu_i}{2}-1/4\right)}\}\,K\{\prod_{i=1}^{r}\,(x_i^{\mu_i+\nu_i-3/2})\,f(x_1^2,\ldots,x_r^2)\,;$$

$$\mu_1-\nu_1\,,\ldots,\,\mu_r-\nu_r\,;\ 2\sqrt{u_1},\ldots,\,2\sqrt{u_r}\}. \qquad \ldots\ (1.12)$$

2. Results Required:

(a) The following integral involving two H-functions of r variables is required in the sequel :

If $H_1\,[\mathbf{Z}]$ be given by (1.2) and

$$H_1^{(1)}\,[\mathbf{Z}] = H_{B'}^{A'}\left[\mathbf{Z}^T\left|\begin{matrix}C'\\ D'\end{matrix}\right.\right],\ \mathbf{Z}^T = \begin{bmatrix}z_1\\ \vdots\\ z_r\end{bmatrix} \qquad \ldots\ (2.1)$$

$$A' \equiv 0,0\ :\ m'_1,\,n'_1\,;\ldots;\,m'_r,\,n'_r$$

$$B' \equiv p',\,q' : p'_1,\,q'_1\,;\ldots\ p'_r,\,q'_r$$

$$C \equiv (a'_j; \rho'_j ,\ldots,\rho_j^{(r)})_{1, p'} : (e'_j , E'_j)_{1, p'_1} ;\ldots; (e_j^{(r)} , E_j^{(r)})_{1, p'_r}$$

$$D \equiv (b'_j, \sigma'_j ,\ldots, \sigma_j^{(r)})_{1, q'} : (f_j ,F'_j)_{1, q'_1} ;\ldots; (f_j^{(r)} , F_j^{(r)})_{1, q'_r}$$

then

$$\int_0^\infty \cdots \int_0^\infty x_1^{-1} \ldots x_r^{-1}\ H_1 [\lambda_1 x_1^{-1} ,\ldots, \lambda_r x_r^{-1}]\ H_1^{(1)} [k_1 x_1 ,\ldots, k_r x_r]$$

$$dx_1 \ldots dx_r = H_{B''}^{A''} \left[\begin{array}{c|c} \lambda_r k_1 & C'' \\ \vdots & \\ \lambda_r k_r & D'' \end{array} \right] \qquad \ldots (2.2)$$

with

$$A'' \equiv 0,0 : m_1 + m'_1 , n_1 + n'_1 ; m_r + m'_r , n_r + n'_r$$

$$B'' \equiv p+p' , q+q' : p_1+p_1 , q_1+q'_1 ;\ldots; p_r+p'_r , q_r+q'_r$$

$$C'' \equiv (a_j; \alpha'_j,\ldots,\alpha_j^{(r)})_{1, p} , (a'_j; \rho'_j,\ldots,\rho_j^{(r)})_{1 . p'} : (c'_j, \gamma'_j)_{1, n_1} ,$$

$$(e'_j, E'_j)_{1, p'_j} , (c'_j , \gamma'_j)_{n_1+1, p_1} ;\ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1, n_r} ,$$

$$(e_j^{(r)}, E_j^{(r)})_{1, p'_r} , (c_j^{(r)}, \gamma_j^{(r)})_{n_r+1, p_r}$$

$$D'' \equiv (b_j; \beta'_j,\ldots,\beta_j^{(r)})_{1, q} , (b'_j; \sigma'_j, \ldots, \sigma_j^{(r)})_{1, q'} : (d'_j, \delta'_j)_{1, m_1} ,$$

$$(f'_j, F'_j)_{1, q'_j} , (d'_j , \delta'_j)_{m_1+1, q_1} ;\ldots; (d_j^{(r)}, \delta_j^{(r)})_{1, m_r} ,$$

$$(f_j^{(r)}, F_j^{(r)})_{1\, q'_r} , (d_j^{(r)} , \delta_j^{(r)})_{m_r+1, q_r} .$$

The conditions of validity of (2.1) are

$$- \min_{1\leqslant j \leqslant n_i} [Re\{ (1-c_j^{(i)}) / \gamma_j^{(i)}\}] - \min_{1\leqslant j \leqslant m'_i} [Re(f_j^{(i)} / F_j^{(i)})] < 0$$

$$\triangleleft \min_{1\leqslant j \leqslant m_i} [Re(d_j^{(i)} / \delta_j^{(i)})] + \min_{1 \leqslant j \leqslant n'_i} [Re\{ (1-e_j^{(i)}) / E_j^{(i)} \}] ,$$

for $i = 1,\ldots,r.$

(b) The following property of multidimensional Mellin convolution is also required :

If $_*$ denotes the rth order Mellin convolution, we have

$$x_1^{\alpha_1}\dots x_r^{\alpha_r}\,[\,\{x_1^{\beta_1}\dots x_r^{\beta_r}\,f(\mathbf{X})\}_*\{x_1^{\gamma_1}\dots x_r^{\gamma_r}\,g(\mathbf{X})\}\,]$$

$$=\{x_1^{\alpha_1+\beta_1}\dots x_r^{\alpha_r+\beta_r}\,f(\mathbf{X})\}_*\{x_1^{\alpha_1+\gamma_1}\dots x_r^{\alpha_r+\gamma_r}\,g(\mathbf{X})\}. \qquad \dots (2.3)$$

(c) If $\hat{f}(\mathbf{U})$ denotes the H-function transform of $f(\mathbf{X})$ as given by (1.7), we have

$$M\{\hat{f}(\mathbf{U})\,;\mathbf{S}\} = M\{f(\mathbf{X})\,;-\mathbf{S}\}\,M\,[\mathbf{X}]\,;\mathbf{S}\} \qquad \dots (2.4)$$

provided that the rth order integrals involved in (2.4) are absolutely convergent. Similarly, if

$$\hat{g}(\mathbf{U}) = \int_0^\infty \dots \int_0^\infty x_1^{-1}\dots x_r^{-1}\,H_1^{(1)}\,[u_1x_1,\dots,u_rx_r]\;g(\mathbf{X})\,dx_1\dots dx_r$$

$$= H^{A',\,\{C'\}}_{B',\,\{D'\}}\,[g(\mathbf{X})\,;\mathbf{U}\} \qquad \dots (2.5)$$

wher $H_1^{(1)}$ [.] is given by (2.1), then

$$M\{\hat{g}(\mathbf{U})\,;\mathbf{S}\} = M\{g(\mathbf{X});-s_1,\dots,-s_r\}\;M\{H_1^{(1)}\,[\mathbf{X}]\,;\mathbf{S}\}, \qquad \dots (2.6)$$

where A', B', C' and D' are given with (2.1).

3. The Main Theorem

Theorem 1. if $\hat{f}(\mathbf{U})$ and $\hat{g}(\mathbf{U})$ denote the multidimensional H–function transforms of the functions $f(\mathbf{X})$ and $g(\mathbf{X})$ as given by equations (1.7) and (2.5), respectively, then

$$\hat{f}(\mathbf{U})_*\,\hat{g}(\mathbf{U}) = H^{A'',\,\{C''\}}_{B'',\,\{D''\}}\,[\,f(\mathbf{X})\,_*\,g\,\mathbf{X})\,;\mathbf{U}\,] \qquad \dots (3.1)$$

where $*$ denotes the multidimensional Mellin convolution as defined in (1.6) (A'', B'', C'', D'', are given with equation (2.2)), provided that each of the involved integrals is absolutely convergent.

Proof : To establish (3.1), we show that the multidimensional Mellin transforms of the left hand side and the right hand side of (3.1) are equal.

The Mellin transform of left hand side of (3.1)

$$= \; M\{\hat{f}(\mathbf{U}) * \hat{g}\,;\mathbf{S}\}$$

$$= \; M\{\hat{f}(\mathbf{U})\,;\mathbf{S}\}\; M\{\hat{g}(\mathbf{U})\,;\mathbf{S}\}$$

$$= \; M\{H_1\,[\mathbf{U}]\,;\mathbf{S}\}\, M\{H^{(1)}_{1}\,[\mathbf{U}]\,;\mathbf{S}\}\; F(-\mathbf{S})\; G(-\mathbf{S}), \qquad \dots (3.2)$$

in view of equation (1.5), (2 4) and (2.6).

Next, let I_2 denote the Mellin transform of the right hand side of (3.1), then we have

$$I_2 \;=\; \int_0^\infty \!\dots\! \int_0^\infty u_1^{s_1-1} \dots u_r^{s_r-1} \Big\{ \int_0^\infty \cdot \int_0^\infty x_1^{-1} \dots x_1^{-1}$$
$$H^{A''}_{B''}\left[\begin{matrix} u_1x_1 \\ \vdots \\ u_rx_r \end{matrix}\,\middle|\,\begin{matrix} C'' \\ \\ D'' \end{matrix}\right] (f(\mathbf{X}) * g(\mathbf{X}))\, dx_1 \dots dx_r \Big\}\, du_1 \dots bu_r\,.$$

On changing the orber of integrations (which is valid due to the absolute convergence of the integrals involved) and on putting $u_i x_i = y_i$, $i = 1,\dots,r$, we get

$$I_2 = F(-\mathbf{S})\; G(-\mathbf{S}) \int_0^\infty \!\dots\! \int_0^\infty y_1^{s_1-1} \dots\, y_r^{s_r-1}\; H^{A''}_{B''}\left[\, \mathbf{Y}^T \,\middle|\, \begin{matrix} C'' \\ D'' \end{matrix} \right] dy_1 \dots dy_r\,.$$

Now on substituting for $H^{A''}_{B''}$ [.] from (2.2), making another change of order of integrations on the right hand side and then putting $y_i = x_i t_i$, $i = 1,\dots,r$, in the inner integral, we arrive at the right hand side of (3.2). This establishes the desired result.

If we take $m'_1 = m_1$, $n'_1 = n_1$ etc. and make suitable changes in the parameters of the multidimensional $H^{(1)}_1$ [**X**] function in Theorem 1, it yields the following, on simplification:

Corollary 1

$$H^{A,\ \{C\}}_{B,\ \{D\}}\,[f(\mathbf{X})\,;\,\mathbf{U}]\,*\,H^{A,\ \{C_4\}}_{B,\ \{D_4\}}\,[g(\mathbf{X})\,;\,\mathbf{U}]$$

$$=\pi^{M_1}\,2^{M_2}\,H^{A,\ \{C_5\}}_{B,\ \{D_5\}}\,[\,(f*g)(x^2_1,\ldots,x^2_r)\,;\,2^{-R_1}\sqrt{u_1},\ldots,2^{-R_r}\sqrt{u_r}]\ldots(3.3)$$

where

$$M_1 \equiv \sum_{i=1}^{r}\left(m_i+n_i-\frac{p_i+q_i}{2}\right)-\frac{p+q}{2},$$

$$M_2 \equiv \sum_{i=1}^{r}(m_i+n_i-p_i)+2\left(\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j+\sum_{i=1}^{r}\sum_{j=1}^{q_i}c^{(i)}_j-\sum_{i=}^{r}\sum_{j=1}^{q_i}d^{(i)}_j\right)-p,$$

$$R_i \equiv \sum_{j=1}^{p}\alpha^{(i)}_j-\sum_{j=1}^{q}\beta^{(i)}_j+\sum_{j=1}^{p_i}\gamma^{(i)}_j-\sum_{j=1}^{q_i}\delta^{(i)}_j,\ i=1,\ldots,r,$$

$$C_4 \equiv (a_j+1/2\ ;\ \alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p} : (c'_j+1/2,\gamma'_j)_{1,p_1};\ldots;$$

$$(c^{(r)}_j+1/2\,,\,\gamma^{(r)}_j)_{1,p_r}\,.$$

$$D_4 \equiv b_j+1/2;\ \beta'_j,\ldots,\beta^{(r)}_j)_{1,q}\,(d'_j+1/2,\,\delta'_j)_{1,q_1},\ldots,$$

$$(d^{(r)}_j+1/2\,,\,\delta^{(r)}_j)_{1,q_r}$$

$$C_5 \equiv 2a_j\!:\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}\!:(2c'_j,\gamma'_j)_{1,p_1};\ldots;(2c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r},$$

$$D_5 = 2b_j : \beta'_j,\ldots,\beta^{(r)}_j)_{1,q} : (2d'_j,\delta'_j)_{1,q_1};\ldots;(2d^{(r)}_j,\delta^{(r)}_j)_{1,q_r}$$

provided that the various rth order integrals involved are absolutely convergent.

4. Applications

1. If we reduce the multidimensional H-function transform to the multidimensional Laplace transforms, then, in view of (1.9) and (2.12), (3.1) yields

$$\left(u_1^{\varepsilon_1} \ldots u_r^{\varepsilon_r} L\{x_1^{\varepsilon_1-1} \ldots x_r^{\varepsilon_r-1} f(\mathbf{X})\,;\,\mathbf{U}\}\right)$$

$$* \left(u_1^{\nu_1} \ldots u_r^{\nu_r} L\{x_1^{\nu_1-1} \ldots x_r^{\nu_r-1} g(\mathbf{X})\,;\,\mathbf{U}\}\right)$$

$$= 2^{3r/2} \prod_{i=1}^{r} \left(u_i^{\frac{\varepsilon_i+\nu_i}{2} - 1/4}\right) K\{x_1^{\varepsilon_1+\nu_1-3/2} \ldots x_r^{\varepsilon_r+\nu_r-3/2} (f_* g)(x_1^2, \ldots, x_r^2);$$

$$\varepsilon_1-\nu_1, \ldots, \varepsilon_r-\nu_r\,;\, 2\sqrt{u_1}, \ldots, 2\sqrt{u_r}\}. \qquad \ldots (4.1)$$

2. If, in (3.3) we take

$$g(x_1, \ldots, x_r) = x_1^{\lambda_1} \ldots x_r^{\lambda_r}\, \delta(x_1-1) \ldots \delta(x_r-1),$$

where $\delta(x)$ denotes the Dirac delta function, then, after a little simplification, it yields the following result:

$$\int_0^\infty \cdots \int_0^\infty \hat{f}\left(\frac{u_1}{x_1}, \ldots, \frac{u_r}{x_r}\right) H_B^A \left[\begin{array}{c|c} x_1 & C_4 \\ \vdots & \\ x_r & D_4 \end{array}\right] \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r}$$

$$\pi^{M_1}\, 2^{M_2}\, H_{B,\,\{D_5\}}^{A,\,\{C_5\}} \left[f(x_1^2, \ldots, x_r^2)\,;\, 2^{-R_1}\sqrt{u_1}, \ldots, 2^{-R_r}\sqrt{u_r} \right]. \qquad \ldots (4.2)$$

If, in (4.2). we take $r=2$ and reduce the double H-function transform to the double Laplace transform, we arrive at the known result [1, p.97, *eqn.* (51)], after a little simplification.

The results established in § 3 are capable of yielding many other results, but these are not recorded here for lack of space.

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SOME FINITE INTEGRALS INVOLVING THE PRODUCT OF THE MULTIVARIABLE H-FUNCTION AND A GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

In this paper, we establish six finite ictegrals involving the product of Jacobi polynomials,a general class of polynomials, and the multivariable H-function. The results established here are quite general in nature and a large number of (known and new) integrals can be obtained by specializing the parameters suitably of the various functions involved in them.

1. INTRODUCTION

Srivastava and Panda [5] defined the mulivariable H- function as (see also [6] and [7])

$$H\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = H^{0,\varepsilon\,:\,(M',\,N')\,;\ldots,\,(M^{(r)},\,N^{(r)})}_{A,C:\,(B',\,D')\,;\ldots;\,(B^{(r)},\,D^{(r)})}$$

$$\left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}:\phi^{(r)}]; \\ [(c):C',\ldots,C^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\ \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right)$$

$$= \frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r} V(s_1,\ldots,s_r)\; U_1(s_1)\ldots U_r(s_r)\; z_1^{s_1}\ldots z_r^{s_r}\; ds_1\ldots ds_r \qquad \ldots (1.1)$$

where

$$V(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{\varepsilon}\Gamma\left(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_i\right)}{\prod_{j=\varepsilon+1}^{A}\Gamma\left(a_j-\sum_{i=1}^{r}\theta_j^{(i)}s_i\right)\prod_{j=1}^{C}\Gamma\left(1-c_j+\sum_{i=1}^{r}C_j^{(i)}s_i\right)} \qquad \ldots (1.2)$$

and

$$U_i(s_i) = \frac{\prod\limits_{j=1}^{M^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_i) \prod\limits_{j=1}^{N^{(i)}} \Gamma(1-b_j^{(i)} + \phi_j^{(i)} s_i)}{\prod\limits_{j+M^{(i)}+1}^{D^{(i)}} \Gamma(1-d_j^{(i)} + \delta_j^{(i)} s_i) \prod\limits_{j=N^{(i)}+1}^{B^{(i)}} \Gamma(b_j^{(i)} - \phi_j^{(i)} s_i)}$$

$$i=1,\ldots,r \quad \ldots (1.3)$$

The multiple integral in (1.1) converges absolutely if

$$|\arg(z_i)| < \frac{1}{2} T_i \pi, \; i=1,\ldots,r \qquad \ldots (1.4)$$

where

$$T_i = -\sum_{j=\varepsilon+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{N^{(i)}} \phi_j^{(i)} - \sum_{j=N^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} C_j^{(i)} + \sum_{j=1}^{M^{(i)}} \delta_j^{(i)}$$

$$- \sum_{j=M^{(i)}+1}^{D^{(i)}} \delta^{(i)} > 0, \; i=1,\ldots,r \qquad \ldots (1.5)$$

We shall require the following definition ([8], p. 1, Eq. (1)).

$$S_n^m [x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{nk}, \; x^k, \; n=0,1,2,\ldots \qquad \ldots (1\ 6)$$

where m is arbitrary positive integer and the coefficients $A_{n,k}(n,k \geqslant 0)$ are arbitrary constants, real complex.

by suitably specialzing the coefficients $A_{n,k}$, the polynomials $S_n^m [x$ can be reduced to the well-known classical orthogonal polynomials such as Jacobi, Hermite, Legendre, Tchebycheff, and Laguerre polynomials, etc.

The following notations and known results will be used throughout this paper :

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1, \text{ is } n = 0 \\ \alpha(\alpha+1)\ldots(\alpha+n-1), \; \forall \; n \in \{1,2,3,\ldots\} \end{cases} \qquad \ldots (1.7)$$

Next, for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [4, p. 254, Eb. (1)], we have

$$P_k^{(\alpha,\beta)}(t+\rho)\,P_k^{(\alpha,\beta)}(t-\rho) = \frac{(-1)^k(1+\alpha)_k(1+\beta)_k}{(k!)^2}$$

$$\sum_{r=0}^{k}(-k)_r\,\frac{(1+\alpha+\beta+k)_r}{(1+\alpha)_r\,(1+\beta)_r}\;P_r^{(\alpha,\alpha)}(x,t^r \qquad \text{... (1.8)}$$

$$\rho^k\,P_k^{(\alpha,\alpha)}\left(\frac{1-xt}{\rho}\right) = \frac{(1+a)_k}{k!}\sum_{r=0}^{k}\frac{{}_r(-k)}{(1+a)_r}\,P_r^{(\alpha,\alpha)}(x)\,t^r \qquad \text{... (1.9)}$$

$$\frac{1}{\rho}(1-t+\rho)^{-\alpha}(1+t+\rho)^{\beta} = 2^{-\alpha-\beta}\sum_{r=0}^{\infty}P_r^{(\alpha,\beta)}(x)\,t^r \qquad \text{... (1.10)}$$

In each of the formulas (1.8), (1.9) and (1.10), and throughout this paper, $\rho = (1-2xt+t^2)^{1/2}$. Formulas (1.8), (1.9) and (1.10) can be found, for example in [1, p. 945] and [2, p. 172], respectively.

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\eta}\,P_t^{(\alpha,\beta)}(x)\,S_n^{m}\left[(1+x)^{\sigma}\right]\,H\begin{pmatrix} z_1(1+x)^{h_1} \\ \vdots \\ z_r(1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{\eta+\alpha\quad\sigma k+1}\;\Gamma(a+t+1)\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}\,A_{n,k}$$

$$H^{0,\quad \varepsilon+2:\,(M',N');\ldots;(M^{(r)},N^{(r)})}_{A+2,C+2:\,(B',D');\ldots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[-\eta-\sigma k:h_1,\ldots,h_r], \\ [(c):\,C',\ldots,C^{(r)}],\end{matrix}\right.$$

$$\begin{matrix}[\beta-\eta-\sigma k:\;h_1,\ldots,h\;]:\,[(a):\theta',\,\cdot\,,\theta^{(r)}]: \\ [\beta\;\eta-\sigma k+t:h_1,\ldots,h_r],[-a-\eta-\sigma k-t-1:\,h_1,\cdot\cdot,h_r]:\end{matrix}$$

$$\left.\begin{matrix}[(b'):\,\phi'];\ldots;[(b^{(r)}):\,\phi^{(r)}]; \\ [(d\;):\,\delta'];\ldots;[(d^{(r)}):\,\delta^{(r)}];\end{matrix}\;\begin{matrix}2^{h1}\,z_1 \\ \vdots \\ 2^{hr}\,z_r\end{matrix}\right) \qquad \text{... (1.11)}$$

where $Re(\alpha) > -1$, $Re(\eta) > -1$, $\sigma > 0$, $h_i > 0$, $i = 1,\ldots,r$

and $|\,arg(z_i)\,| < \frac{1}{2}\,T_i\,\pi$, $T_i > 0$ and

$Re\,[\eta + \sum_{i=1}^{r} h_i\, d_j^{(i)} / \delta_j^{(i)}] > -1$, m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n,k \geqslant 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} (1-x)^{\eta} (1+x)^{\sigma} P_t^{(\alpha,\beta)}(x)\, S_n^m [(1-x)^u (1+x)^v]\, H\begin{pmatrix} z_1(1-x)^{h1}(1+x)^{k1} \\ \vdots \\ z_r(1-x)^{hr}(1+x)^{kr} \end{pmatrix}$$

$$= 2^{\eta+\sigma+(u+v)k+1} \sum_{k=0}^{[n/m]} \sum_{r=0}^{t} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(-t)_r\,(\alpha+\beta+t+1)_r}{(\alpha+1)_r \quad r!}$$

$$H^{0,\ \ \varepsilon+2\,:\,(M',\,N')\,;\ldots;\,(M^{(r)},\,N^{(r)})}_{A+2,\,C+1\,:\,(B',\,D')\,;\ldots;\,(B^{(r)},\,D^{(r)})} \left(\begin{matrix} [-\sigma-vk;\,k_1,\ldots,k_r], \\ [(c):\,C',\ldots,C^{(r)}], \end{matrix}\right.$$

$$\begin{matrix} [-r-\eta-uk:h_1,\ldots,h_r]\,:\,[(a)\,:\,\theta',\ldots\,\theta^{(r)}]\,: \\ [-1-r-\eta-\sigma-(u+v)k:\,h_1+k_1,\ldots,h_r+k_r]: \end{matrix}$$

$$\left.\begin{matrix} [(b'):\,\phi']\,;\ldots;\,[(b^{(r)})\,:\,\phi^{(r)}]; \\ [(d'):\,\delta']\,;\ldots;\,[(d^{(r)})\,:\,\delta^{(r)}]; \end{matrix} \begin{matrix} z_1 2^{h1+k1} \\ \vdots \\ z_r 2^{hr+kr} \end{matrix}\right) \qquad \ldots\ (1.12)$$

where

$Re(\eta) > -1$, $Re(\sigma) - 1$, $u > 0$, $v > 0$, $h_i > 0$, $k_i > 0$,

$i = 1,\ldots,r$, $|\arg(z_i)| < \frac{1}{2} T_i\,\pi$, $T_i > 0$,

$Re(\eta + \sum_{i=1}^{r} h_i\, d_j^{(i)} / \delta_j^{(i)}] > -1$, $Re[\sigma + \sum_{i=1}^{r} k_i\, d_j^{(i)} / \delta_j^{(i)}] > -1$

and m is an arbitrary positive integer and the coefficients $A_{n,k}(n,k \geqslant 0)$ are arbitrary constants, real or complex. The integral (1.12) can be evaluated by making use of a known result [3, p. 284, Eq. (2)]. In fact, (1.12) is an obvious special case of much more general results given by Srivastava and Singh [9, p. 172, Eq. (3.2)].

2. MAIN INTEGRALS

The following integrals have been established in this paper :

$$\int_{-1}^{1} (1+x)^{\eta} (-x)^{\alpha} P_{s}^{(\alpha,\beta)} (t+\rho) P_{s}^{(\alpha,\beta)} (t-\rho) S_{n}^{m} [(1+x)^{\sigma}] H\begin{pmatrix} z_1(1+x)^{h1} \\ \vdots \\ z_r(1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{\eta+\alpha+\sigma k+1} \frac{(-1)^{s}\,\Gamma(1+\alpha+s)\,(1+\beta)_{s}}{(s!)^{2}} \sum_{k=0}^{[n/m]} \sum_{R=0}^{s} \frac{(-n)_{mk}\, A_{n,k}}{k!}$$

$$(-s)_{R} \frac{(1+\alpha+\beta+s)_{R}}{(1+\beta)_{R}} t^{R}\, H^{0,\quad \epsilon+2\,:\,(M',N');\ldots;(M^{(r)},N^{(r)})}_{A+2,C+2\,:\,(B',D');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix} [-\eta-\sigma k:h_1,\ldots,h_r],\ [\beta-\eta-\sigma k:h_1,\ldots,h_r] : [(a): \theta',\ldots,\theta^{(r)}] : \\ [(c) : C',\ldots,C^{(r)}],\ [\beta-\eta-\sigma k+R: h_1,\ldots,h_r],\ [-\alpha-\eta\ \sigma k-R-1]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'): \phi'] ;\ldots; [(b^{(r)}) : \phi^{(r)}] ; \\ [h_1,\ldots,h_r] : [(d'): \delta],\ldots,[d^{(r)}) : \delta^{(r)}]; \end{matrix} \begin{matrix} z_1 2^{h_1} \\ \vdots \\ z_r 2^{h_r} \end{matrix} \right) \qquad \ldots (2.1)$$

where $Re(\eta) > -1, Re(\alpha) > 1, \sigma > 0, h_i > 0, i = 1,\ldots,r$

and $| arg(z_i) | < \frac{1}{2} T_i\pi, T_i > 0, Re[\eta + \sum_{i=1}^{r} h_i\, d_j^{(j)} / \delta_j^{(i)}] > -1$

and m is an arbitrary posittive integer and the coefficients $A_{n,k}(n,k \geqslant 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} \rho^{s} (1+x)^{\eta} (1-x)^{\alpha} P_{s}^{(\alpha,\alpha)} \left(\frac{1-xt}{\rho} \right) S_{n}^{m} [(1+x)^{\sigma}] H \begin{pmatrix} z_1(1+x)^{h_1} \\ \vdots \\ z_r(1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{\eta+\alpha+\sigma k+1} \frac{\Gamma(1+\alpha+s)}{s!} \sum_{R=0}^{s} \sum_{k=0}^{[n/m]} (-s)_{R}\, t^{R} \frac{(-n)_{mk}}{k!} A_{n,k}$$

$$H^{0,\quad \epsilon+2\,:\,(M',N');\ldots;(M^{(r)},N^{(r)})}_{A+2,C+2\,:\,(B',D');\ldots;(B^{(r)},D^{(r)})} \left(\begin{matrix} [-\eta-\sigma k:h_1,\ldots,h_r], \\ [(c): C',\ldots,C^{(r)}], \end{matrix} \right.$$

$$\begin{matrix} [\alpha-\eta-\sigma k:h_1,\ldots,h_r] : [(a): \theta',\ldots,\theta^{(r)}] : [(b') : \phi'] ;\ldots; \\ [\alpha-\eta-\sigma k+R:h_1,\ldots,h_r],[-\alpha-\eta-\sigma k-R-1:h_1,\ldots,h_r] : [(d) : \delta'] ;\ldots; \end{matrix}$$

$$\left. \begin{matrix} [(b^{(r)}) : \phi^{(r)}] ; \\ [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \begin{matrix} z_1 2^{h_1} \\ \vdots \\ z_r 2^{h_r} \end{matrix} \right) \qquad \ldots (2.2)$$

where $Re(\eta) > -1, Re(\alpha) > -1, \sigma > 0, h_i > 0, i = 1, \ldots, r,$

$| \arg(z_i) | > \frac{1}{2} T_i \pi, T_i > 0, Re(\eta + \sum_{i=1}^{r} h_i d_j^{(i)} / \delta_j^{(i)}] > 1$

and m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n,k \geqslant 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} \frac{1}{\rho} (1-t+\rho)^{-\alpha} (1+t+\rho)^{-\beta} (1+x)^{\eta} (1-x)^{\alpha} S_{n}^{m} [1+x)^{\sigma}]$$

$$H\begin{pmatrix} z_1(1+x)^{h_1} \\ \vdots \\ z_r(1+x)^{h_r} \end{pmatrix} dx = 2^{\eta+\sigma k-\beta-1} \sum_{R=0}^{\infty} \sum_{k=0}^{[n/m]} \Gamma(a+R+1)\, t^{R} \frac{(-n)_{mk} A_{n,k}}{k!} .$$

$$. \; H^{0,\ \ \varepsilon+2\,:(M',N');\ldots;(M^{(r)},N^{(r)})}_{A+2,C+2\,:(B',D');\ldots;(B^{(r)},D^{(r)})} \left(\begin{array}{l} [-\eta-\sigma k : h_1, \ldots, h_r], \\ [(c) : C', \ldots, C^{(r)}], \end{array} \right.$$

$$\begin{array}{l} [\beta-\eta-\sigma k : h_1, \ldots, h_r] : [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] \quad ; \ldots ; \\ [\beta-\eta-\sigma k : k+R_i, h_1, \ldots, h_r], [-a-\eta-\sigma k-R-1 : h_1, \ldots, h_r] : [(d') : \delta'] ; \ldots ; \end{array}$$

$$\left. \begin{array}{l} [(b^{(r)}) : \phi^{(r)}]; \\ [(d^{(r)}) : \delta^{(r)}]; \end{array} \begin{array}{c} z_1 2^{h_1} \\ \vdots \\ z_r 2^{h_r} \end{array} \right) \qquad \ldots (2.3)$$

where $Re(\eta) > -1, Re(a) > -1, \sigma > 0, Re\, \eta + \sum_{i=1}^{r} h_i d_j^{(i)} / \delta_j^{(i)}] > -1,$

$i = 1, \ldots, r, | \arg(z_i) | < \frac{1}{2} T_i \pi, T_i > 0, h_i > 0,$

and m is an arbitrary positive integer and the coefficients $A_{n,k}(n,k \geqslant 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} (1-x)^{\eta} (1+x)^{\sigma} P_{s}^{(\alpha,\beta)} (t+\rho)\, P_{s}^{(\alpha,\beta)} (t-\rho)\, S_{n}^{m} (1-x)^{u} [(1+x)^{v}]$$

$$. \; H\begin{pmatrix} z_1 (1-x)^{h1} \\ \vdots \\ z_r (1-x)^{h_r} \end{pmatrix}$$

$$= 2^{\eta+\sigma+(u+v)k+1} \frac{(-1)^s\Gamma(1+\alpha+s)\Gamma(1+\beta+s)}{(s!)^2} \,.$$

$$\sum_{R=0}^{s} \sum_{k=0}^{[n/m]} \sum_{l=0}^{R} \frac{(-s)_R(1+\alpha+\beta+s)_R}{\Gamma(1+\alpha+R)\Gamma(1+\beta+R)} \frac{(-n)_{mk}\, A_{n,k}}{k!} \frac{(-R)_l(\alpha+\beta+R+1)_l}{(1+\alpha)_l\; l!}\; t^R$$

$$H^{0,\;\;\varepsilon+2\;:(M',\,N')\;;\dots;\;(M^{(r)},\,N^{(r)})}_{A+2,C+1:(B',\,D')\;;\dots;\;(B^{(r)},\,D^{(r)})} \left(\begin{matrix}[-\sigma-vk:k_1,\dots,k_r], \\ [(c):C',\dots,C^{(r)}],\end{matrix}\right.$$

$$\begin{matrix}[-\eta-uk-l:h_1,\dots,h_r] : [(a):\theta',\dots,\theta^{(r)}] : [(b'):\phi'] ,\dots, \\ [-\eta-\sigma-(u+v)k-l-1:h_1+k_1,\dots,h_r+k_r] : [(d'):\delta'] ,\dots,\end{matrix}$$

$$\left.\begin{matrix}[(b^{(r)}):\phi^{(r)}] \;; \\ [(d^{(r)}):\delta^{(r)}] \;;\end{matrix} \begin{matrix} z_1 2^{h_1+k_1} \\ \vdots \\ z_r 2^{h_r+k_r}\end{matrix}\right) \qquad \dots (2.4)$$

wheae $Re(\eta)>-1, Re(\sigma)>-1$, $u>0$, $v>0$, $Re[\eta + \sum_{i=1}^{r} h_i\, d_j^{(i)} / \delta_j^{(i)}] > -1$,

$Re[\sigma + \sum_{i=1}^{r} k_i\, d_j^{(i)} / \delta_j^{(i)}] > -1$, $i = 1,\dots,r$, $|\arg(z_i)| < \frac{1}{2} T_i \pi$,

$T_i > 0$, $h_i > 0$, $k_i > 0$, and m is an arbitrary positive integer and the coefficients $A_{n,k}(n,k \geqslant 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} (1-x)^{\eta}\; (1+x)^{\sigma}\; \rho^s\, P_s^{(\alpha,\alpha)}\left(\frac{1-xt}{\rho}\right) S_n^m\, [(1-x)^u\, (1+x)^v]$$

$$H\left(\begin{matrix} z_1(1-x)^{h_1}(1+x)^{k_1} \\ \vdots \\ z_r(1-x^{h_r}(1+x)^{k_r}\end{matrix}\right) dx$$

$$= 2^{\eta+\sigma+(u+v)k+1} \frac{\Gamma(1+\alpha+s)}{s!} \sum_{R=0}^{s} \sum_{k=0}^{[n/m]} \sum_{l=0}^{R} \frac{(-s)_R\, t^R}{\Gamma(1+\alpha+R)} \frac{(-n)_{mk}}{k!}\, A_{n,k}$$

$$\cdot \frac{(-R)_l\,(2\alpha+R+1)_l}{(\alpha+1)_l\; l!}\; H^{0,\;\;\varepsilon+2\;:(M',\,N')\;;\dots;\;(M^{(r)},\,N^{(r)})}_{A+2,\;C+1\;:(B',\,D')\;;\dots;\;(B^{(r)},\,D^{(r)})}$$

$$\left(\begin{matrix}[-\sigma-vk: k_1,\ldots,k_r], [-\eta-uk-l: h_1,\ldots,h_r], [(a): \theta',\ldots,\theta^{(r)}] : \\ [(c): C',\ldots,C^{(r)}]. [\eta-\sigma-(u+v)k-l-1 : h_1+k_1,\ldots,h_r+k_r]: \end{matrix}\right.$$

$$\left.\begin{matrix}[(b'): \phi'] ;\ldots; [(b^{(r)}): \delta^{(r)}] ; \\ [(d'): \delta'] ;\ldots; [(d^{(r)}): \delta^{(r)}] ; \end{matrix} \begin{matrix} z_1 2^{h_1+k_1} \\ \vdots \\ z_r 2^{h_r+k_r}\end{matrix}\right) \quad \text{.. (2.5)}$$

where $Re(\eta) > -1$, $Re(\sigma) > -1$, $u > 0$, $v > 0$,

$Re[\eta + \sum_{i=1}^{r} h_i d_j^{(i)} / \delta_i^{(i)} [> -1$, $Re[\sigma + \sum_{i=1}^{r} k_i d_j^{(i)} / \delta_j^{(i)}] > -1$,

$i = 1,\ldots,r$, $| arg(z_i) | < \frac{1}{2} T_i\pi$, $T_i > 0$, $h_i > 0$, $k_i > 0$, and m is an arbitrary positive integer and the coefficients $A_{n,k}(n,k \geqslant 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} (1-x)^{\eta} (1+x)^{\sigma} \frac{1}{2} (1-t+\rho)^{-\alpha} (1+t+\rho)^{-\beta} S_n^m [(1-x)^u (1+x)^v].$$

$$. H\begin{pmatrix} z_1(1-x)^{h_1} (1+x)^{k_1} \\ \vdots \\ z_r(1-x)^{h_r} (1+x)^{k_r}\end{pmatrix} dx$$

$$= 2^{-\alpha-\beta+\eta+\sigma+(u+v)k+1} \sum_{R=0}^{\infty} \sum_{k=0}^{[n/m]} \sum_{l=0}^{R} t^R \frac{(-n)_{mk}}{k!} A_{n,k} \cdot \frac{(-R)_l (\alpha+\beta+R+1)_l}{(\alpha+1)_l\, l!}$$

$$. H^{0,\varepsilon+2 :(M',N');\ldots;(M^{(r)},N^{(r)})}_{A+2,C+2 :(B',D');\ldots;(B^{(r)},D^{(r)})} \left(\begin{matrix}[-\sigma-vk:k_1,\ldots,k_r], \\ [(c) : C',\ldots,C^{(r)}]\end{matrix}\right.$$

$$\begin{matrix}[-\eta-uk-l :h_1,\ldots,h_r] : [(a): \theta',\ldots,\theta^{(r)}] : [(b'): \phi'] ;\ldots; \\ [-\eta-\sigma-(u+v)k-l-1:h_1+k_1,\ldots,h_r+k_r]: [(d'): \delta'] ;\ldots;\end{matrix}$$

$$\left.\begin{matrix}[(b^{(r)}) : \phi^{(r)}] ; \\ [(d^{(r)}) : \delta^{(r)}] ;\end{matrix} \begin{matrix} z_1 2^{h_1+k_1} \\ \vdots \\ z_r 2^{h_r+k_r}\end{matrix}\right) \quad \text{.. (2.6)}$$

where $Re(\eta) > -1$, $Re(\sigma) > -1$, $u > 0$, $v > 0$,

$$Re[\eta + \sum_{i=1}^{r} h_i \, d_j^{(i)} / \delta_j^{(i)}] > -1, Re[\sigma + \sum_{i=1}^{r} k_i \, d_j^{(i)} / \delta_j^{(i)}] > -1,$$

$$h_i > 0, k_i > 0, i = 1, \ldots, r, \; |\arg(z_i)| < \frac{1}{2} T_i \pi, T_i > 0$$

and m is an arbitrary constant and the coefficients $A_{n,k} (n,k \geqslant 0)$ are arbitrary constants, real or complex.

Proof: To establish (2.1), multiply both the sides of (1.8) by

$$(1+x)^{\eta} (1-x)^{\alpha} \, S_n^m \, [(1+x)^{\sigma}] \, H \begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{pmatrix} \text{ and}$$

integrating both sines with respect to x between the limits -1 to 1, we get

$$\int_{-1}^{1} (1+x)^{\eta} (1-x)^{\alpha} \; P_s^{(\alpha, \beta)} (t+\rho) \; P_s^{(\alpha, \beta)} (t-\rho) \; S_n^m \, [(1+x)^{\sigma}]$$

$$H \begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{pmatrix} dx$$

$$= \int_{-1}^{1} \frac{(-1)^s \, (1+\alpha)_s \, (1+\beta)_s}{(s!)^2} \sum_{R=0}^{s} \frac{(-s)_R \, (1+\alpha+\beta+s)_R}{(1+\alpha)_R \, (1+\beta)_R}$$

$$. \; P_R^{(\alpha, \beta)} (x) \, t^R \, (1+x)^{\eta} (1-x)^{\alpha} \, S_n^m \, [(1+x)^{\sigma}] \, H \begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_1 (1+x)^{h_r} \end{pmatrix} dx \quad \ldots(2.7)$$

Now, express the $S_n^m [x]$ function on the left hand side of (2.7) by its series (1.6), then interchange the order of integration and summation in the right hand side of (2.7), which is justified due to the absolute convergent of the integral involved in the process and then evaluate the inner integral with the help of (1.11).

Finally, interpreting by virtue of (1.1), we get the desired result.

The proofs of the formulas (2.2) to (2.6) can be developed by preceeding on similar lines with the help of the result (1.9) through (1.12), respectively.

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A MULTIDIMENSIONAL GENERATING FUNCTION

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ABSTRACT

A generalisation of Bailey's transform is used to deduce a multivariate generating function with certain arbitrary coefficiehts. This is then applied to obtain generating functions for the Appell biorthogonal polynomials in terms of Bessel functions and the Tratnik biorthogonal polynomials in terms of confluent hypergeometric functions. Finally, a new generating function for polynomials of one variable is given and applied to obtain a new result appertaining to the Jacobi polynomials.

1. Introduction and definitions.

One of the most general transformations of single series was geven by Bailey [2] and Slater [7, p. 58]. This was later extended to multidimensional series by Exton [4, p. 139] in the form now given.

If

$$\beta_{m_1,\ldots,m_n} = \sum_{p_1,\ldots,p_n=0}^{m_1,\ldots,m_n} a_{p_1,\ldots,p_n} u_{m_1-p_1,\ldots,m_n-p_n} v_{m_1+p_1,\ldots,m_n+p_n}$$

and ... (1.1)

$$\gamma_{m_1,\ldots,m_n} = \sum_{p_1=m_1,\ldots,p_n=m_n}^{\infty} \delta_{p_1,\ldots,p_n} u_{p_1-m_1,\ldots,p_n-m_n}$$
$$v_{p_1+m_1,\ldots,p_n+m_n}, \qquad \ldots (1.2)$$

then $\Sigma\, a_{m_1,\ldots,m_n} \gamma_{m_1,\ldots,m_n} = \Sigma\, \beta_{m_1,\ldots,m_n} \delta_{m_1,\ldots,m_n}$. ... (1.3)

Here and elsewhere in this study, the symbol Σ without further qualification denotes the multiple summation with respect to the indicies of summation $m_1,\dots,m_n$ over all non-negative integer values. The quantities α, δ, u and v are functions of $p_1,\dots,p_n$ only and any questions of convergence must be settled individually as appropriate.

The Pochhammer symbol (a, n) is frequently used in the theory of hypergeometric functions and is given by

$$(a,n) = a(a+1) \dots (a+n-1) = \Gamma(a+n) / \Gamma(a);\ (a,0) = 1 \quad \dots (1.4)$$

and (a) denotes the sequence of parameters $a_1,\dots,a_A$. The single hypergeometric function of general order is given by

$$ {}_AF_B \begin{bmatrix} a_1,\dots,a_A; & \\ & x \\ b_1,\dots,b_B; & \end{bmatrix} = {}_AF_B[{(a); \atop (b);} x] = \sum_{m=0}^{\infty} \frac{((a),m)x^m}{((b),m)m!} . \quad \dots (1.5)$$

Karlsson's [5] generalised Kampé de Fériet function is represented as

$$F^{A:B}_{C:D} \begin{bmatrix} (a):(b_1);\dots;(b_n); & \\ & x_1,\dots,x_n \\ (c):(d_1);\dots;(d_n); & \end{bmatrix} = \sum \frac{((a),M)((b_1),m_1)\dots((b_n),m_n)x_1^{m_1}\dots x_n^{m_n}}{((c),M)((d_1),m_1)\dots((d_n),m_n)m_1!\dots m_n!}, \quad \dots (1.6)$$

where $M = m_1 + \dots + m_n$ *etc.*

The Lauricella function of the first kind is also employed:

$$F_A^{(n)}(a,b_1,\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n) = F^{1:1}_{0:1} \begin{bmatrix} a:b_1;\dots;b_n; & \\ & x_1,\dots,x_n \\ -c_1;\dots;c_n; & \end{bmatrix} . \quad \dots (1.7)$$

See Lauricella [6].

Any values of the parameters for which any expression does not make sense are tacitly excluded.

2. The Generalised Generating Function.

Theorem I. If A, D, U and V are arbitrary coefficients, then

$$U(0) \sum \frac{A(m_1,\dots,m_n)D(M.V.2M)}{m_1!\dots m_n!} (x_1t_1)^{m_1} \dots (x_{n-1}t_{n-1})^{m_{n-1}}$$

$$(-x_n)^{m_n} (t_1 + \dots + t_{n-1})^{m_n}$$

$$\sum \frac{D(M)t_1^{m_1} \,..\, t_{n-1}^{m_{n-1}}(-t_1-..-t_{n-1})^{m_n}}{m_1!\ldots m_n!} \sum_{p_1,\ldots,p_n=0}^{m_1,\ldots,m_n} A(p_1,\ldots,p_n)$$

$$\frac{U(M-P)V(M+P)\,(-m_1,p_1)\,..\,(-m_n,p_n)\,(-x_1)^{p_1}\ldots(-x_n)^{p_n}}{p_1!\ldots p_n!} \quad \ldots (2.1)$$

Proof . In (1.1) and (1.2) put

$$u_{m_1,..,m_n} = U(M)\,/\,(m_1!\,..\,m_n!), \quad \ldots (2.2)$$

$$v_{m_1,..,m_n} = V(M), \quad \ldots (2.3)$$

$$\delta_{m_1,..,m_n} = D(M)\,t_1^{m_1}\,..\,t_{n-1}^{m_{n-1}}\,(-t_1-..-t_{n-1})^{m_n} \quad \ldots (2.4)$$

and

$$\alpha_{m_1,..,m_n} = A(m_1,..,m_n)\,x_1^{m_1}\,..\,x_n^{m_n}\,/\,(m_1!\,..\,m_n!). \quad \ldots (2.5)$$

Then

$$\beta_{m_1,..,m_n} = \frac{1}{m_1!\,..\,m_n!} \sum_{p_1,..,p_n=0}^{m_1,..,m_n}$$

$$\frac{A(p_1,..,p_n)U(M-P)V(M+P)\,(-m_1,p_1)\,..\,(-m_n,p_n)}{p_1!\,..\,p_n!}\,(-x_1)^{p_1}\,..\,(-x_n)^{p_n} \quad (2.6)$$

and

$$\gamma_{m_1,..,m_n} = \sum_{p_1,..,p_n=0}^{\infty}$$

$$\frac{D(M+P)\;U(P)V(P+2M)\,t_1^{m_1+p_1}\,..\,t_{n-1}^{m_{n-1}+p_{n-1}}\,(-t_1-..-t_{n-1})^{m_n+p_n}}{p_1!\ldots p_n!}. \quad \ldots (2.7)$$

Now, as *a* simple consequence of the binomial theorem

$$\sum_{m,n=0}^{\infty} \frac{F(m+n)}{m!n!} \frac{x^m y^n}{} = \sum_{m=0}^{\infty} \frac{F(m)}{m!}\,(x+y)^m.$$

See Srivastava [8] .

Hence $\gamma_{m_1,..,m_n} = t_1^{m_1} \,..\, t_{n-1}^{m_{n-1}}(-t_1 ..-t_{n-1})^{m_n}$

$$\times \sum_{p=0}^{\infty} \frac{D(M+P)U(P)V(2M+P)}{p!} \, (t_1 + .. t_{n-1} - t_1 - ..-t_{n-1})^p$$

$$= t_1^{m_1} \,..\, t_{n-1}^{m_{n-1}} \, (-t_1 - ..-t_{n-1})^{m_n} \, U(0) \, D(M) V(2M) \qquad ...(2.9)$$

since the argument of the power series in p is zero. Theorem I follows immediately if (2.6) and (2.9) are substituted into (1.3).

3. Generating Functions for Appell's Biorthogonal Polynomials E and F.

Biorthogonal polynomials in two variables E and F due to Appell (Appell et Kampé de Fériet [1, Chapter VI] may readily be generalised to several variables in the form

$$E_{m_1,..,m_n}(c_1,..,c_n;x_1,..,x_n) = {}^{(n)}_{A}(C+M,-m_1,..,-m_n;c_1,..,c_n;x_1,..,x_n) \qquad ... (3.1)$$

and

$$F_{m_1,..,m_n}(c_1,..,c_n,x_1,..,x_n) = (1-x)^M \, F_A^{(n)} \, (-M,-m_1,..,-m_n;c_1,..,c_n;x_1,..,x_n). \qquad ... (3.2)$$

In the case of the polynomials E, put $U(M) = 1$, $V(M) = (C,M)$ and

$$A(m_1,..,m_n) = 1 \,/\, [(c_1,m_1) \,...\, (c_n,m_n)] \qquad ... (3.3)$$

into Theorem I. We then have

$$\sum \frac{D(M)(C,M)\, t_1^{m_1} \,..\, t_{n-1}^{m_{n-1}} (-t_1 - .. -t_{n-1})^{m_n}}{m_1!...m_n!} \, F_A^{(n)} \, (C+M,-m_1,..,-m_n; c_1,..,c_n;-x_1,..,-x_n)$$

$$= \sum \frac{D(M)(C,2M)(x_1t_1)^{m_1}..(x_{n-1}t_{n-1})^{m_{n-1}}[-x_n(t_1 + .. + t_{n-1})]^{m_n}}{(c_1,m_1)..(c_n,m_n)\; m_1!...m_n!} \qquad ... (3.4)$$

If x_1 is replaced by $-x_1$ and $D(M)$ by $1/(C,2M)$, (3.4) takes the form

$$\sum \frac{(C,M)\, t_1^{m_1} \ldots t_{n-1}^{m_{n-1}} (-t_1-\ldots-t_{n-1})^{m_n}}{(C,2M)\, m_1! \ldots m_n!} F_{m_1,\ldots,m_n}(c_1,\ldots,c_n;x_1,\ldots x_n)$$

$$= {}_0F_1(-;c_1;-x_1t_1) \ldots {}_0F_1(-;c_{n-1};-x_{n-1}t_{n-1})\, {}_0F_1(-;c_n;x_n[t_1+\ldots+t_{n-1}])$$

$$\Gamma(c_1-1) \ldots \Gamma(c_n-1)\, (x_1t_1)^{1-c_1} \ldots (x_{n-1}t_{n-1})^{1-c_{n-1}} [x_n(t_1+\ldots+t_{n-1})]^{1-c_n}$$

$$\times J_{c_1-1}[2\sqrt{(x_1t_1)}] \ldots J_{c_{n-1}-1}[2\sqrt{(x_{n-1}t_{n-1})}]\, I_{c_n-1}\{2\sqrt{(x_n(t_1+\ldots+t_{n-1})}]\}. \tag{3.5}$$

Similarly, in the case of the polynomials F, put $A(m_1,\ldots,m_n)=1/[(c_1,m_1)\ldots(c_n,m_n)]$, $U(M)=1/(1,M)$, $V(M)=1$ and $D(M)=1$ and replace x_i by $x_i/(X-1)$, when we have

$$\sum \frac{t_1^{m_1} \ldots t_{n-1}^{m_{n-1}} (-t_1-\ldots-t_{n-1})^{m_n}}{(1,M)\, m_1! \ldots m_n!\, (1-X)^M} F_{m_1,\ldots,m_n}(c_1,\ldots,c_n;x_1,\ldots,x_n)$$

$$= \Gamma(c_1-1) \ldots \Gamma(c_n-1)\, [x_1t_1/(X-1)]^{1-c_1} \ldots [x_{n-1}t_{n-1}/(X-1)]^{1-c_{n-1}}$$

$$\left[\frac{x_n(t_1+\ldots+t_{n-1})}{X-1}\right]^{1-c_n}$$

$$\times I_{c_1-1}\{2\sqrt{x_1t_1/(X-1)]}\} \ldots I_{c_{n-1}-1}\{2\sqrt{[x_{n-1}t_{n-1}/(X-1)]}\}$$

$$\times I_{c_n-1}\{2\sqrt{[x_n(t_1+\ldots+t_{n-1})/(X-1)]}\}. \qquad \ldots(3.6)$$

4. The Tratnik Biorthogonal Polynomiuls.

Biorthogonal polynomials generalising the continuous Hahn polynomials have been introduced and discussed by Tratnik [9]. They are given by the expressions

$$P^{a,b,c,d}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \frac{i^M (A+d.M)(a_1+b_1,m_1) \ldots (a_n+b_n,m_n)}{m_1! \ldots m_n!}$$

$$\times\ F^{1:2}_{1:1}\left[\begin{matrix} A+B+c+d+M-1 : a_1+ix_1,-m_1;\,..\,;a_n+ix_n,-m_n; \\ A+d \qquad ; \quad a_1+b_1\ ;\,..\,;\quad a_n+b_n\ ; \end{matrix}\ 1,\,..\,,1\right] \qquad \ldots(4.1)$$

and their counterparts

$$Q^{a,b,c,d}_{m_1,\,..\,,m_n}(x_1,\,..\,,x_n) = \frac{(-i)^M\,(c+iX,M)\,(a_1+b_1,m_1)\,..\,(a_n+b_n,m_n)}{m_1!\ldots m_n!}$$

$$\times\ F^{1:2}_{1:1}\left[\begin{matrix} 1-c-d-M\ : b_1-ix_1,-m_1;\,..\,;b_n-ix_n,-m_n; \\ 1-c-iX-M;\quad a_1+b_1\quad ;\,..\,;\quad a_n+b_n\ ; \end{matrix}\ 1,\quad ,1\right]. \qquad \ldots(4.2)$$

Firstly, let $A(m_1,\quad ,m_n) = \dfrac{(a_1+ix_1,m_1)\,..\,(a_n+ix_n,m_n)}{(a_1+b_1,m_1)\,..\,(a_n+b_n,m_n)(A+d,M)}$,
$U(M)=1$,
$V(M) = (A+B+c+d-1,M)$ and $x_1 = \,..\, = x_n = 1$. This gives

$$\sum \frac{D(M)\,(A+B+c+d-1,M)\ t_1^{m_1}\,..\,t_{n-1}^{m_{n-1}}\,(-t_1-\,..\,-t_{n-1})^{m_n}\,(-i)^M}{(A+b,M)}$$

$$P^{a,b\ c,d}_{m_1,\,..\,,m_n}(x_1,\ \ ,x_n)$$

$$=\Sigma\ \frac{D(M)\left(\dfrac{A+B+c+d-1}{2},M\right)\left(\dfrac{A+B+c+d}{2},M\right)(a_1+ix_1,m_1)\,..\,(a_n+ix_n,m_n)}{(a_1+b_1,m_1)\,..\,(a_n+b_n,m_n)\ m_1!\,..\,m_n!}$$

$$\times\ (-4t_1)^{m_1}\,..\,(-4t_{n-1})^{m_{n-1}}(t_1+\,..\,t_n+{}_{-1})^{m_n}\ 4^{m_n}. \qquad \ldots(4.3)$$

Now let $D(M) = 1/[(\dfrac{A+B+c+d-1}{2},M)\,(\dfrac{A+B+c+d}{2},M)]$, when (4.3) assumes the simpler form

$$\sum \frac{(A+B+c+d-1,M)\ t_1^{m_1}\,..\,t_{n-1}^{m_{n-1}}\,(-t_1-\,..\,-t_{n-1})^{m_n}\,(-i)^M}{(A+d,M)\left(\dfrac{A+B+c+d-1}{2},M\right)\left(\dfrac{A+B+c+d}{2},M\right)}$$

$$P^{a\ b,c,d}_{m_1,\,..\,,m_n}(x_1,\,..\,,x_n) = {}_1F_1\,(a_1+ix_1;a_1+b_1;-4t_1)\,..\,{}_1F_1\,(a_{n-1}+ix_{n-1};$$

$$a_{n-1}+b_{n-1};-4t_{n-1}) \times {}_1F_1\,(a_n+ix_n;a_n+b_n;4t_{n-1}). \qquad \ldots(4.4)$$

Similarly, in the case of the polynomials Q, let $V(M) = 1$, $D(M) = 1$, $U(M) = (c+iX,M) / (c+d,M)$ and

$$A(m_1, \ldots, m_n) = \frac{(b_1-ix_1,m_1) \ldots (b_n-ix_n,m_n)}{(a_1+b_1,m_1) \ldots (a_n+b_n,m_n)} .$$

This gives the result

$$\sum \frac{t_1^{m_1} \ldots t_{n-1}^{m_{n-1}} (-t_1 - \ldots - t_{n-1})^{m_n} \, i^M}{(c+d,M)(a_1+b_1,m_1) \ldots (a_n+b_n,m_n)} \, Q^{a,b,c,d}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$$

$$= {}_1F_1(b_1-ix_1; a_1+b_1; -t_1) \ldots {}_1F_1(b_{n-1}-ix_{n-1}; a_{n-1}+b_{n-1}; -t_{n-1})$$

$$\times \, {}_1F_1(b_n-ix_n; a_n+b_n; t_1+\ldots+t_{n-1}). \qquad \ldots (4.5)$$

5. A Generating Function for a Class of Polynomials of One Variable.

In theorem I, put $n = 2$, $t_1 = t$, so as to obtain the formula

$$U(0) \sum_{m_1,m_2=0}^{\infty} \frac{A(m_1,m_2) D(m_1+m_2) V(2m_1+2m_2)}{m_1! m_2!} (x_1 t)^{m_1} (-x_2 t)^{m_2}$$

$$= \sum_{m_1,m_2=0}^{\infty} \frac{D(m_1+m_2) \, t^{m_1+m_2} (-1)^{m_2}}{m_1! m_2!} \times \sum_{p_1,p_2=0}^{m_1,m_2} A(p_1,p_2)$$

$$\frac{U(m_1+m_2-p_1-p_2) \, V(m_1+m_2+p_1+p_2) \, (-m_1,p_1)(-m_2,p_2)}{p_1! \, p_2!}$$

$$\times \, (-x_1)^{p_1} (-x_2)^{p_2} . \qquad \ldots (5.1)$$

$$U(0) \sum_{m=0}^{\infty} \frac{A(m,0) D(m) V(2m) (xt)^m}{m!} = \sum_{m_1,m_2=0}^{\infty} \frac{D(m_1+m_2) t^{m_1+m_2} (-1)^{m_2}}{m_1! m_2!}$$

$$\sum_{p=0}^{m_1} \frac{A(p,0) U(m_1+m_2-p) V(m_1+m_2+p) (-m_1,p)(-x)^p}{p!} \qquad \ldots (5.2)$$

The coefficient $D(m)$ may be selected arbitrarily so that the form of the function on the left is at our disposal.

As an example, let $A(p,0) = 1 / (a+1,p)$, $U(P)=i$, $V(P)=(a+b+1,p)$. A class of new generating functions for the Jacobi polynomials

$$P_m^{(a,b)}(z) = \frac{(a+1,m)}{m!}\ {}_2F_1\,(a+b+1+m,-m;a+1;1/2-z/2)$$ Erdélyi *et al.* [3, p. 170])

then follows if x is replaced by $z/2-1/2$. This takes the form

$$\sum_{m_1,m_2=0}^{\infty} \frac{D(m_1+m_2\,(a+b+1,m_1+m_2)\,t^{m_1+m_2}\,(-1)^{m_2}}{(a+1,m_1)m_2!}\ P_{m_1}^{(a,b)}(z)$$

$$= \sum_{m_1=0}^{\infty} \frac{D(m)\,([a+b+1]/2,m)([a+b]/2+1,m)}{(a+1,m)\ m!}\,[2t(z-1)]^m . \qquad \dots (5.3)$$

If we let $D(m) = \dfrac{(a+1,m)}{([a+b+1/2],m)\,([a+b]/2+1,m)}$, the right-hand member of (5.3) becomes an exponential function. Hence,

$$\sum_{m_1,m_2=1}^{\infty} \frac{(a+1,m_1+m_2)\,(a+b+1,m_1+m_2)\ t^{m_1+m_2}\,(-1)^{m_2}}{(a+1,m_1)\,([a+b+1]/2,m_1+m_2)([a+b]/2+1,m_1+m_2)m_2!}$$

$$P_{m_1}^{(a,b)}(z) = \exp\,[2t(z-1)] . \qquad \dots (5.4)$$

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A FIXD POINT THEOREM FOR A PAIR OF EXPANSIVE MAPS

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In a recent paper the author [1] established several fixed point theorems for a pair of surjective expansive maps. In this note we prove a theorem for a pair of expansive maps, and demonstrate that the first two results of [1] follow as special cases.

Theorem. *Let f,g be surjective selfmaps of a complete metric space (X,d). Suppose there exists a constant $a > 1$ such that*

(1) $d(fx,gy) \geqslant a\,d(x,y)$

for each x,y in $\{\overline{x_n}\}$ satisfying either $x \in f^{-1}y$ or $y \in g^{-1}x$, *where $\{x_n\}$ is defined by $x_0 \in X$, $x_{2n+1} \in f^{-1}x_{2n}$, $x_{2n+2} \in g^{-1}x_{2n+1}$. Then f and g have a unique common fixed point.*

Proof. The proof of Theorem 1 of [1] only involves points in the closure of $\{x_n\}$. Therefore that proof applies, unchanged, to this theorem.

Corollary 1. ([1, Theorem 1]). *Let f,g be surjective selfmaps of a complete metric space (X,d). Suppose there exists a constant $a > 1$ such that*

(2) $d(fx,gy) \geqslant a\,d(x,y)$

for each x,y in X. Then f and g have a unique common fixed point.

Since (2) is true for all points x,y in X, it is surely true for those points in the closure of $\{x_n\}$,

Corollary 2. ([1, Theorem 2]). *Let f,g be surjective selfmaps of a complete metric space (X d). Suppose there exists nonnegative functions* p,q,r,s,t *satisfying*

$$(3)\quad \inf_{x,y\in X}\ (p(x,y)+q(x,y)+t(x,y)) > 1,$$

$$(4)\quad \inf_{x,y\in X}\ \{1-q(x,y)+r(x,y),\ (1-p(x,y)+s(x,y))\} > 0,$$

$$(5)\quad \sup_{x,y\in X}\ \{p(x,y),\ q(x,y)\} > 1,$$

and

$$(6)\quad d(fx,gy) \geqslant p(x,y)\, d(x,fx) + q(x,y)\, d(y,gy) + r(x,y)\, d(x,gy) + s(x,y)\,(dy,fx) + t(x,y)\, d(x,y)$$

for all x,y in X, $x=y$. Then f and g have a unique common fixed point.

Proof. Set $y = fx$ in (6) to get

$$\begin{aligned} d(fx,gfx) &\geqslant pd(x,fx) + qd(fx,gfx) + rd(x,gfx) + td(x,fx) \\ &\geqslant pd(x,fx)+qd(fx,gfx)+r[d(x,fx)-d(fx,gfx)]+td(x,fx), \end{aligned}$$

where p,q,r,s,t are evaluated at (x, fx). We then obtain

$$d(fx,gfx) \geqslant ad(x,fx),$$

where

$$a = \inf_{x,y\in X} \frac{p+t+r}{1-q+r} > 1$$

from (2).

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ON INTEGRALS INVOLVING THE MULTIVARIABLE H-FUNCTION

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ABSTRACT

Four integrals involving the products of two general classes of polynomials of several variables with the multivariable H-function are established here.

1. INTRODUCTION

(a) The multivariable H-function

The multivariable H-function of Srivastava and Panda [7] is defined and represented as follows:

$$H[z_1,\ldots,z_r] = H^{0,\epsilon\,:\,m_1,n_1\,;\ldots;\,m_r,n_r}_{A,C:\,p_1,q_1\,;\,\ldots;\,p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right|$$

$$\begin{matrix} (a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,A} : (c'_j,\gamma'_j)_{1,\,p_1};\ldots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,\,p_r} \\ (b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,C} : (d'_j,\delta'_j)_{1,\,p_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,\,q_r} \end{matrix}$$

$$= \frac{1}{(2\pi\omega))^r}\int_{L_1}\cdots\int_{L_r}\cdots\psi(\xi_1,\ldots,\xi_r)\,\phi_1(\xi_1)\ldots\phi_r(\xi_r)$$

$$z_1^{\xi_1}\ldots z_r^{\xi_r}\,d\xi_1\ldots d\xi_r,\ \omega=\sqrt{-1}. \qquad \ldots(1.1)$$

For the convergence and existence conditions and other details of the multivariable H-function, we refer to Srivastava *et al.* [5, pp. 251-253, eqs. (C.2) - (C.8)].

When $\epsilon = 0$ in (1.1), we denote the right-hand side of (1.1) by $H_1[z_1,\dots,z_r]$. Also, the appearance of the asterisk (*) indicates that the parameters at those places are the same as the parameters of the multivariable H-function in the left-hand side of (1.1) at the corresponding places.

If we take $r = 2$, the multvariable H-function defined in (1.1) reduces to the H-function of two variables [5,p 82,eq. (6.1.1)]

(b) General classes of polynomials

A general class of polynomials of one variable, studied recently by Srivastava, is defined and represnted as follows [2,p.1,eq. (1)] :

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k\,;\ n,m = 0,1,\dots;\ m \neq 0, \qquad \dots (1.2)$$

where $A_{n,k}$ are arbitrary constants.

We here study the following generalisation of this general class of polynomials due also to Srivastava [3,p.185,eq. (7)] :

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[x_1,\dots,x_r] = \sum_{h_1=0}^{[n_1/m_1]} \dots \sum_{h_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 h_1}}{h_1!} \dots \frac{(-n_r)_{m_r h_r}}{h_r!}$$

$$A(h_1,\dots,h_r)\, x_1^{h_1} \dots x_r^{h_r}, \qquad \dots (1.3)$$

where, for $i = 1,\dots,r$, $n_i, m_i = 0,1,2,\dots$; $m_i \neq 0$; the coefficients $A(h_1,\dots,h_1)$ being arbitrary constants, real or complex.

Another general class of polynomials of r variables studied here is defined and represented by Srivastava and Garg in the following form [6,p.686,eq. (1.4)] :

$$T_n^{m_1,\ldots,m_r}[x_1,\ldots,x_r]$$

$$= \sum_{h_1,\ldots,h_r=0}^{M \leqslant n} \left[(-n)_M \; B(n,h_1,\ldots,h_r) \prod_{i=1}^{r} \left(\frac{x_i^{h_i}}{h_i!} \right) \right], \qquad \ldots (1.4)$$

with $M = \sum_{i=1}^{r} m_i h_i$, $(m_i \geqslant 1,\ i = 1,\ldots,r)$ and $B(n,h_1,\ldots,h_r)$ being arbitrary constants, real or complex.

By suitably specialising the coefficients $A_{n,k}$, $A(h_1,\ldots,h_r)$ and $B(n,h_1,\ldots,h_r)$, the polynomials defined in (1.2), (1.3) and (1.4), reduce to a number of known polynomials. To just cite an example, if in (1.4), we take $m_i = 1$, $(i = 1,\ldots,r)$ and

$$B(n,h_1,\ldots,h_r) = \frac{\prod_{i=1}^{r} (\beta_i)_{h_i}}{(\alpha)_{M'}},\ M' = \sum_{i=1}^{r} h_i ,$$

it yields the following relation :

$$T_n^{1,\ldots,1}[x_1,\ldots,x_r] = \sum_{h_1,\ldots,h_r=0}^{M' \leqslant n} \left[\frac{(-n)_{M'}}{(\alpha)_{M'}} \prod_{i=1}^{r} \left((\beta_i)_{h_i} \frac{x_i^{h_i}}{h_i!} \right) \right]$$

$$= F_D^{(r)} [-n, \beta_1,\ldots,\beta_r \,;\, \alpha ;\, x_1,\ldots,x_r], \qquad \ldots (1.5)$$

where $F_D^{(r)}[-n, \beta_1,\ldots,\beta_r \,;\, \alpha \,;\, x_1,\ldots,x_r]$ is the Lauricella polynomial [4,p.382, eq. (25)]. In what follows, we use

$$\Omega_i = \min_{1 \leqslant j \leqslant m_i} Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right),\ i = 1,\ldots,r . \qquad \ldots (1.6)$$

2. The Main Integrals

For $\lambda_i > \sigma_i > 0$, $\mu_i > \omega_i > 0$ and

$$\left[Re(\rho, + \sum_{i=1}^{r} \mu_i \Omega_i \right] > \left[Re(v) + \sum_{i=1}^{r} \omega_i \Omega_i \right] > 0, \; i = 1,\ldots,r,$$

we have:

$$\int_0^\infty x^{v-1} (a+x)^{-\rho} S_{N_1,\ldots,N_r}^{M_1,\ldots,M_r} \left[y_1 x^{\sigma_1} (a+x)^{-\lambda_1}, \ldots, \lambda_r \; x^{\sigma_r} (a+x)^{-\lambda_r} \right]$$

$$. \; H_1 \left[z_1 x^{\omega_1} (a+x)^{-\mu_1}, \ldots, z_r x^{\omega_r} (a+x)^{-\mu_r} \right] dx$$

$$= a^{v-\rho} \sum_{h_1=0}^{[N_1/M_1]} \cdots \sum_{h_r=0}^{[N_r/M_r]} \left(\prod_{i=1}^{r} \left[\frac{(-N_i)_{M_i h_i}}{h_i\,!} (y_i \, a^{\sigma_i - \lambda_i})^{h_i} \right] \right.$$

$$\left. A(h_1,\ldots,h_r) \; . \; H^{0\;\;,2\;\;:*;\ldots;*}_{A+2,C+1:*;\ldots;*} \left[\begin{matrix} z_1 \; a^{\omega_1 - \mu_1} \\ \vdots \\ z_r \; a^{\omega_r - \mu_r} \end{matrix} \middle| \begin{matrix} E \\ \\ F \end{matrix} \right] \right) ; \qquad \ldots (2.1)$$

$$\int_0^\infty x^{v-1} (a+x)^{-\rho} \, T_N^{M_1,\ldots,M_r} \left[y_1 x^{\sigma_1} (a+x)^{-\lambda_1}, \ldots, y_r x^{\sigma_r} (a+x)^{\lambda_r} \right]$$

$$. \; H_1 \left[z_1 x^{\omega_1} (a+x)^{-\mu_1}, \ldots, z_r \; x^{\omega_r} (a+x)^{-\mu_r} \right] dx$$

$$= a^{v-\rho} \sum_{h_1,\ldots,h_r=0}^{M \leqslant N} \left((-N)_M \prod_{i=1}^{r} \left[\frac{(y_i \, a^{\sigma_i - \lambda_i})}{h_i\,!} \right] B(N,h_i,\ldots,h_r) \right.$$

$$\left. . \; H^{0\;\;,2\;\;:*;\ldots;*}_{A+2,C+1:*;\ldots;*} \left[\begin{matrix} z_1 \; a^{\omega_1 - \mu_1} \\ \vdots \\ z_r \; a^{\omega_r - \mu_r} \end{matrix} \middle| \begin{matrix} E \\ \\ F \end{matrix} \right] \right) , \qquad \ldots (2.2)$$

where $M = \sum_{i=1}^{r} M_i h_i$ and

$$E : (1-\nu - \sum_{i=1}^{r} \sigma_i h_i \; ; \; \omega_1,\ldots,\omega_r) \, , (1-\rho+\nu- \sum_{i=1}^{r} (\lambda_i-\sigma_i)h_i \; ; \; \mu_1-\omega_1,\ldots,\mu_r-\omega_r),$$

$$(a_j \; ; \; \alpha'_j \, ,\ldots, \alpha_j^{(r)} \,)_{1,A} \quad *:;\ldots;*$$

$$F : (1- \delta - \sum_{i=1}^{r} \lambda_i h_i \; ; \; \mu_1,\ldots,\mu_r) \, . \, (b_j; \; \beta'_j \, ,\ldots,\beta_j^{(r)} \,)_{1,C} \quad : * \; ;\ldots; * \, .$$

3. Multiple Integral Analogues

Let $\theta = \sum_{i=1}^{r} \eta_i \, x_i^{l_i}$. For $j = 1,\ldots,r$, let

$$X_j = x_1^{\zeta_1^{(j)}} \ldots \, x_r^{\zeta_r^{(r)}} \; \theta^{\omega_j} (a+\theta)^{-\mu_j} \; , X'_j = \theta^{\sigma_j} (a+\theta)^{-\lambda_j} \; ,$$

$$\tau_j = \omega_j + \sum_{i=1}^{r} \frac{\zeta_i^{(j)}}{l_i} \, , \, Z_j = z_j \prod_{i=1}^{r} \eta_i \left(\frac{-\zeta_i^{(i)}}{l_i} \right)$$

$$U = \sum_{i=1}^{r} \frac{\nu_i}{l_i} \text{ and } \phi(\eta_1,\ldots,\eta_r) = \prod_{i=1}^{r} \frac{\eta_i^{(\nu_i/l_i)}}{l_i} \, .$$

Then, for $j = 1,\ldots,r \quad \min\limits_{1\leqslant i \leqslant r} \; (\eta_i, \, l_i, \, \zeta_i^{(j)} \; Re(\nu_i) \,) > 0, \; \mu_j > \tau_i > \omega_j > 0,$

$$\lambda_j > \sigma_j > 0 \text{ and } \left[Re(\rho) + \sum_{j=1}^{r} \mu_j \, \Omega_j \right] > \left[Re(\nu + U) + \sum_{j=1}^{r} \tau_j \Omega_j \right] > 0,$$

we have :

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} (x_i^{\nu_i-1}) \, \theta^{\nu} \; (a+\theta)^{-\rho} \; S_{N_1,\ldots,N_r}^{M_1,\ldots,M_r} \left[y_1 X'_1 \; ,\ldots, y_r X'_r \right]$$

$$\cdot \; H_1 \, [z_1 X_1 \, ,\ldots, \, z_r X_r] \; dx_1 \ldots dx_r$$

$$= a^{\nu+U-\rho}\ \phi(\eta_1,\dots,\eta_r) \sum_{h_1=0}^{[N_1/M_1]} \cdots \sum_{h_r=0}^{[N_1/M_1]} \left(\prod_{i=1}^{r} \left[\frac{(-N_i)_{M_i h_i}}{h_i!} \left(y_i a^{\sigma_i-\lambda_i}\right)^{h_i} \right] \right.$$

$$\left. .\ A(h_1,\dots,h_r)\ H^{0\ ,\ r+2\ :*;\dots;*}_{A+r+2,\ C+2\ :*;\dots;*} \left[\begin{matrix} z_1\ a^{\tau_1-\mu_1} \\ \vdots \\ z_r\ a^{\tau_r-\mu_r} \end{matrix} \middle| \begin{matrix} P \\ \\ Q \end{matrix} \right] \right); \qquad \dots (3.1)$$

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} (x_i^{\nu_i-1})\ \theta^{\nu}\ (a+\theta)^{-\rho}\ T_N^{M_1,\dots,M_r}\ [y_1 X_1',\dots,y_r X_r']$$

$$.\ \ H_1\ [z_1 X_1,\dots,z_r X_r]\ dx_1 \dots dx_r$$

$$= a^{\nu+U-\rho}\ \phi(\eta_1,\dots,\eta_r) \sum_{h_1,\dots,h_r=0}^{M\leqslant N} \left((-N)_M \prod_{i=1}^{r} \left[\frac{\left(y_i a^{\sigma_i-\lambda_i}\right)^{h_i}}{h_i\ !} \right] \right.$$

$$\left. B(N,h_1,\dots,h_r)\ H^{0\ \ \ ,\ r+2\ :*;\dots;*}_{A+r+2,\ C+2\ :*;\dots;*} \left[\begin{matrix} Z_1\ a^{\tau_1-\mu_1} \\ \vdots \\ Z_r\ a^{\tau_r-\mu_r} \end{matrix} \middle| \begin{matrix} P \\ \\ Q \end{matrix} \right] \right), \qquad \dots (3.2)$$

where $M = \sum_{i=1}^{r} M_i h_i$ and

$$P : \left(1-\frac{\nu_i}{l_i}\ ;\ \frac{\zeta_i'}{l_i},\dots,\frac{\zeta_i^{(r)}}{l_i}\right)_{1,r},\ \left(1-\nu-U-\sum_{i=1}^{r} \sigma_i h_i\ ;\ \tau_1,\dots,\tau_r\right),$$

$$\left(1-\rho+\nu+U-\sum_{i=1}^{r} (\lambda_i-\sigma_i) h_i\ ;\ \mu_1-\tau_1,\dots,\mu_r-\tau_r\right),\ \left(a_j;\ \alpha_j',\dots,\alpha_j^{(r)}\right)_{1,A}\ \ *\ ;\dots;\ *$$

$$Q : \left(1-U\ ;\tau_1-\omega_1,\dots,\tau_r-\omega_r\right),\ \left(1-\rho-\sum_{i=1}^{r} \lambda_i h_i\ ;\ \mu_1,\dots,\mu_r\right),\ \left(b_j;\ \beta_j',\dots,\beta_j^{(r)}\right)_{1,C}$$

$$:\ *\ ;\dots;\ *.$$

Outline of proofs. To establish (2.1), we denote the L.H.S. of (2.1) by I. To evaluate I, we first replace the multivariable H-function by[1]

Mellin–Barnes contour integrals with the help of (1.1), then use (1.3) to express $S_{N_1,\dots,N_r}^{M_1,\dots,M_r}$ [.] in its series form, change the order of integration and summation (which is justified under the stated conditions), evaluate the inner most x–integral with the help of [1,p.72,eq.(3.1.16)], and then interpret the resulting Mellin–Barnes contour integrals by virtue of (1.1) to arrive at the desired result (2.1). The integral in (2.2) is established by using the definition of the general class of polynomials $T_N^{M_1,\dots,M_r}$ [.] with the help of (1.4) and proceeding on similar lines as indicated above for establishing (2.1).

To establish (3.1), we ler

$$\mathrm{I}_1 = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{r} (x_i^{\nu_i - 1}) f(\theta) H_1 [z_1 X_1, \dots, z_r X_r] \; dx_1 \dots dx_r,$$

$$\theta = \sum_{i=1}^{r} \eta_i \, x_i^{l_i} \, .$$

We next replace the multivariable H–function H_1 [.], occurring in I_1, by its Mellin–Barnes contour integrals with the help of (1.1), change the order of resulting muliple integrals and proceed as in [8] to get,

$$I_1 = \frac{\phi(\eta_1, \dots, \eta_r)}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \, \phi_1(\xi_1) \dots \phi_r(\xi_r) \, z_1^{\xi_1} \dots Z_r^{\xi_r}$$

$$\frac{\prod_{i=1}^{r} \Gamma\left(\frac{\nu_i}{l_i} + \sum_{i=1}^{r} \frac{\zeta_i^{(j)}}{l_i} \xi_j \right)}{\Gamma\left(U + \sum_{j=1}^{r} (\tau_j - \omega_j) \xi_j \right)} \left[\int_0^\infty z^{U + \sum_{j=1}^{r} (\tau_j - \omega_j) \xi_j - 1} (a+z)^{-\sum_{j=1}^{r} \mu_j \xi_j} f(z) dz \right]$$

$$d\xi_1 \dots d\xi_r.$$

We next set

$$f(z) = z^{\nu} \; (a+z)^{-\rho} \; S_{N_1,\ldots,N_r}^{M_1,\ldots,M_r}\left[\, y_1 z^{\sigma_1} (a+z)^{-\lambda_1},\ldots,y_r z^{\sigma_r} (a+z)^{-\lambda_r} \right]$$

in (3.3), use (1.3) to expresss $S_{N_1,\ldots,N_r}^{M_1,\ldots,M_r}[.]$ in its series form, change the order of integration and sumation (which is justified under the stated conditions,, evaluate the z-integral with the help of [1,p.72,eq. (3.1.16)], and then finally interpret the resulting Mellin-Barnes contour integrals by virtue of (1.1) to arrive at the desired result in (3.1). The result in (3.2) is similarly established on setting

$$f(z) = z^{\nu} (a+\tau)^{-\rho} \; T_N^{M_1,\ldots,M_r}[y_1 z^{\sigma_1} (a+z)^{-\lambda_1} \;,\ldots,\; y_r z^{\sigma_r} (a+z)^{-\lambda_r}\,]$$

in (3.3) and using (1.4) therein.

4. Special Cases

If, in (2.2), we take $M_i = 1$, $(i = 1,\ldots,r)$ and also take

$$B(N,h_1,\ldots,h_r) = \frac{\prod_{i=1}^{r}(\beta_i')_{h_i}}{(\alpha)_{M'}} \;, \; (M' = \sum_{i=1}^{r} h_i),$$

we get the following result involving the Lauricella polynomials [4,p. 382, eq.(25)]:

$$\int_0^{\infty} x^{\nu\ 1} (a+x)^{-\rho} \; F_D^{(r)}\Big[-N, \; \beta_1',\ldots,\beta_r' \; ; \; \alpha; \; y_1 x^{\sigma_1} (a+x)^{\;\lambda_1}$$

$$,\ldots,y_r x^{\sigma_r} (a+x)^{-\lambda_r}\Big] \quad . \quad H_1 \,[z_1\, x^{\omega_1} (a+x)^{-\mu_1},\ldots. z_r x^{\omega_r} (a+x)^{-\mu_r}\,]\; dx$$

$$= a^{\nu-\rho} \sum_{h_1,\ldots,h_r=0}^{M' \leqslant N} \Big(\frac{(-N)_{M'}}{(\alpha)_{M'}} \prod_{i=1}^{r} \left[\frac{(y_i\, a^{\sigma_i - \lambda_i}\;)^{h_i}}{h_i\;!} \, (\beta_i')_{h_i} \right]$$

$$\cdot\, H^{0,\;2\;:\;*\;;\ldots;\;*}_{A+2,\;C+1\;:\;*\;;\ldots;\;*}\left[\begin{matrix} z_1\, a^{\omega_1-\mu_1} \\ \vdots \\ z_r\, a^{\omega_r-\mu_r} \end{matrix}\,\middle|\,\begin{matrix} E \\ \\ F \end{matrix}\right]\Bigg), \qquad \ldots (4.1)$$

where E,F are given with (2.2). The conditions of validity are same as given for (2.2).

Also, if, in (2.2) and (3.2), we take $y_2 = y_3 = \ldots = y_r = 0$ n make suitable changes therein, these yield the known results [9,pp.66-68,eqs.(2.1), (2.5)]. A number of other particular cases can also be deduced from the integrals established here, but these are not recorded here for lack of space .

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A FINITE DIFFERENCE FORMULA FOR A POLYNOMIAL SYSTEM OF SEVERAL VARIABLES

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ABSTRACT

We give an operational representation for a certain multivariable polynomial set by employing the finite difference operator. Several special cases of the main result (2.4) below are also indicated.

1. INTRODUCTION

In recent years various polynomials and special functions have been defined and studied through their operational representations by several authors (see, for example, Agrawal and Chaubey [1], Srivastava and Singhal [10], Al-Salam [2] and Rahman [7]; (see also Srivastava and Manocha [8], Srivastava and Panda [9].). Differential and difference operators have been frequently used in the literature. Several authors (Khanna [4] and Erdélyi et al. [3]) have established finite difference formulae for polynomial sets of one and two variables. Here we shall give a finite difference formula for the polynomial system $Q^A_n\{(x_n),y\}$ defined by us [6] by means of the generating relation (see also Srivastaava and Manocha [8, Chapter 9]):

$$\sum_{n=0}^{\infty} Q^A_n \{(x_m),y\}\, t^n$$

$$= = exp(vyt^r)\, F_A[\, a,(b_m);\, (c_m);\, \mu_1 x_1 t, \ldots, \mu_m x_m t\,] \qquad \ldots(1\ 1)$$

where the various notations and conventions used are exactly as in [6].

2. THE OPERATIONAL REPRESENTATION

From (1.1) we have

$$Q_n^A\{(x_m),y\} = = (\mu_1\lambda_1)^n \sum_{p_1=0}^{[n/r]} \sum_{p_2=0}^{n-rp_1} \cdots \sum_{p_{m-1}=0}^{n-rp_1-\sum_{i=2}^{m-2} p_i} \frac{(a)_{n-rp_1}}{[(p_{m-1}!)]}$$

$$\frac{[(b_{m-1,1})_{(p_{m-1};1)}](-1)^{n\ rp_1-p_2-\cdots-p_{m-2}}}{[(c_{m-1,1})_{(p_{m-1};1)}](n-rp_1-p_2-\cdots-p_{m-1})!}$$

$$(\nu y/(\mu_1 x_1)^r)^{p_1}\,[(\mu_{m-1,1}\,x_{m-1,1}/\mu_{.}x_1)\,p_{m-1};1]$$

$$\sum_{p_m=0}^{n-rp_1-\sum_{i=2}^{m-1} p_i} \frac{(-1)^{n-rp_1-p_2-\cdots-p_m}(n-rp_1-p_2-\cdots-p_{m-1})!}{p_m!\,(n-rp_1-p_2-\cdots-p_m)!}$$

$$\frac{(b_1)_{n\ rp_1-p_2-\cdots-p_m}\,[(b_m)_{p_m}]}{(c_1)_{n-rp_1-p_2-\cdots p_m}\,[(c_m)_{p_m}]}\;(-\mu_m x_m/\mu_1 x_1)^{p_m} \qquad \ldots (2.1)$$

Now, on writing $n-\alpha_m$, b_m+a_m and $c_m+\alpha_m$ for n, b_m and c_m, respectively; using certain elementary rules of finite difference calculus, viz .

$$\Delta_{\alpha}^{n} f(a) = \sum_{k=0}^{n} (-1)^{n-k}\,{}^nC_k\, f(a+k) \qquad \ldots (2.2)$$

and making usual simplifications, we have

$$Q^{A,b_{m-1}+a_{m-1},\,b_m+\alpha_m}_{n-a_{m-1}-a_m;\,c_{m-1}-\alpha_{m-1},\,c_m-a_m}\{(x_m),y\} = \frac{\Gamma(c_{m-1}+\alpha_{m-1})\Gamma(c_m+\alpha_m)}{\Gamma(b_{m-1}+a_{m-1})\Gamma(b_m+a_m)}$$

$$(-\mu_m x_m)^{-\alpha_m}\,(\mu_{m-1}x_{m-1})^{-a_{m-1}}\,(\mu_1 x_1)^n$$

$$\sum_{p_1=0}^{[(n-\alpha_{m-1}-a_m)/r]}\ \sum_{p_2=0}^{n-\alpha_{m-1}-a_m-rp_1} \cdots \sum_{p_{m-2}=0}^{n-a_{m-1}-a_m-rp_1-\sum_{i=2}^{m-3} p_i} \frac{(a)_{n-\alpha_{m-1}-\alpha_m-rp_1}}{[(p_{m-2}!)]}$$

$$\frac{[(b_{m-2,1})_{(p_{m-2;1})}]}{[(c_{m-2,1})_{(p_{m-2;1})}]\,(n-\alpha_{m-1}-\alpha_m-rp_1-p_2-\dots-p_{m-2})!}$$

$$(\nu y/(\mu_1 x_1)^r)^{p_1}\,[(\mu_{m-2,1}x_{m-2,1}/\mu_1 x_1)\,p_{m-2;1}]$$

$$\Delta_{a_{m-1}}^{(n-\alpha_{m-1}-a_m-rp_1-p_2-\dots-p_{m-2})}\left\{\frac{\Gamma(b_{m-1}+a_{m-1})}{\Gamma(c_{m-1}+a_{m-1})}\,(-\mu_{m-1}x_{m-1}/\mu_1 x_1)^{a_{m-1}}\right.$$

$$\Delta_{a_m}^{(n-\alpha_{m-1}-\alpha_m-rp_1-p_2-\dots-p_{m-2})}\left\{\frac{(b_1)_{n-\alpha_{m-1}-a_m-rp_1-p_2-\dots-p_{m-2}}}{(c_1)_{n-\alpha_{m-1}-\alpha_m-rp_1-p_2-\dots-p_{m-2}}}\right.$$

$$\left.\left.\frac{\Gamma(b_m+\alpha_m)}{\Gamma(c_m+\alpha_m)}\,(-\mu_m x_m/\mu_1 x_1)^{a_m}\right\}\right\} \qquad \dots (2.3)$$

Repeating this process and simplifying once again, we finally have our main result:

$$Q^{A,\,(b_m;_1\,+\,a_m;_1)}_{r(m_1-a_1)+m_2,\,(c_m;_1+a_m;_1)}\,\{(x_m),y\} = \frac{(-1)^{m_2}\,[\Gamma(c_{m,1}+\alpha_{m,1})]}{({}_1m-\alpha_1)!\,[\Gamma(b_{m,1}+a_{m,1})]}$$

$$(\nu y/\mu_1 x_1)^r)^{-\alpha_1}\,(\mu_1 x_1)^{r(m_1-a_1+m_2}\,[\mu_{m,1}x_{m,1}/\mu_1 x_1)^{-\,a_m;_1}]$$

$$\Delta_{\alpha_1}^{(m_1\,-\,a_1)}\left\{\frac{(-1)^{r(m_1-\alpha_1)}\,(m_1-\alpha_1)!\,(a)_{r(m_1-\alpha_1)\,+m_2}}{(r(m_1-a_1)+m_2)!}\,(\nu y/(\mu_1 x_1)^r)^{a_1}\right.$$

$$\Delta_{\alpha_2}^{(r\,(m_1-\alpha_1)+m_2)}\left\{\frac{\Gamma(b_2+a_2)}{\Gamma(c_2+a_2)}\,(-\mu_2 x_2/\mu_1 x_1)^{\alpha_2}\right.$$

$$\Delta_{a_3}^{(r\,(m_1-\alpha_1)\,+m_2)}\left\{\;\dots\;\Delta_{\alpha_m}^{(r\,(m_1-a_1)+m_2}\left\{\frac{(b_1)_{r(m_1-a_1)}}{(c_1)_{r(m_1-\alpha_1)}}\right.\right.$$

$$\left.\left.\frac{\Gamma(b_m+\alpha_m)}{\Gamma(c_m+a_m)}\,(-\mu_m x_m/\mu_1 x_1)^{a_m}\right\}\dots(m-1)\dots\right\} \qquad \dots (2.4)$$

3. APPLICATIONS

We choose to list of following three applications of our main result (2.4) :

1: On setting $\nu=-1$, $r=2$, $y=m=1$, $b_1=c_1$, $x_1=x$, $\mu_1=2/a$ and taking the limit as $a \to \infty$, we have

$$H_{2(m_1-a_1)}(x)=\sqrt{\pi}\,2^{2(m_1-\alpha_1)}\,x^{2n}\,(1/2)_{(m_1-a_1)}$$

$$\Delta_{a_1}^{(m_1-a_1)}\left\{\frac{(-1)^{\alpha_1}}{x^{2a_1}\,\Gamma(m_1-\alpha_1+1/2)}\right\}$$

for the Hermite polynomials.

2: Next, using the conditions $\nu=y=r=m=1$, $a=b_1$, $c_1=1+\alpha$, $\mu_1=-1/a^2$, $x_1=x$ and letting $a\to\infty$, we have

$$L_{m_1-\alpha_1}^{(1+a)}(x) = \frac{(1+a)_{(m_1-\alpha_1)}(-x)^{m_1}}{(m_1-a_1)!}$$

$$\Delta_{\alpha_1}^{(m_1-a_1)}\left\{(-1)^{m_1} \;/\; x^{a_1}\,(1+a)_{(m_1-a_1)}\right\}$$

for Laguerre polynomials.

3: Finally, setting $\nu=y=r=m=1$, $b_1=\alpha+\beta+1$, $c_1=\beta+1$, $\mu_1=1/2a$, $x_1=x-1$, and using the limit as $a\to\infty$, we have

$$P_{m_1-\alpha_1}^{(\beta,a-m_1-a_1)}(x) = \frac{(1+\beta)_{(m_1-\alpha_1)}\,(-1)^{\alpha_1}\,(x-1)^{m_1}}{2^{m_1}\,(m_1-\alpha_1)!}$$

$$\Delta_{a_1}^{(m_1-a_1)}\left\{\frac{(-1)^{(\alpha_1-m_1)}\,(1+\alpha+\beta)_{(m_1-\alpha_1)}\,2^{\alpha_1}}{(1+\beta)_{(m_1-\alpha_1)}\,(x-1)^{\alpha_1}}\right\}$$

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CAUCHY'S PROBLEM FOR PARTIAL DIFERENTIAL EQUATIONS WITH BESSEL-TYPE DIFFERENTIAL OPERATORS

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ABSTRACT

We obtain existence theorems for the solutions of certain linear parabolic equations which contain different Bessel-type differential operators. These operators are infinitesimal generators of semigrops, these being differentiable and analytic. The semigroups are characterized in the early paragraphs. The equations analysed appear as particular cases of the Fokker-Planck equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left[A(x)\frac{\partial}{\partial x}\ [u(x,t)] \right] + \frac{\partial}{\partial x}\left[B(x)\ u(x,t) \right],\ x>0,\ t\in[0,T]$$

when this equation has the most outstanding biological and physical significance.

1. INTRODUTION

The study of uniqueneses classes and the classes where Cauchy's Problem for partial differential systems and equations with Bessel type differential operators remain well posed was initiated by Zitomirskii [18] in 1955. This author obtains a uniqueness class for such systems between the classical functions with temperen growth when he takes into consideration partial differential equations which contain the operator $\triangle_\mu = \frac{d^2}{dx^2} + \frac{2\mu+1}{x}\ \frac{d}{dx}$. Later the research of intial-value problems for systems with the operator $\triangle_\mu$ has been dealt with Kiiprijanov-Kononenko [10]; Krehivshiv-Mattiuck [11]; Brzezinski [2]; among other authors.

W. Lee [12] was the first author who realized a study with a methodology similar to the one carried out by Gelfand–Shilov in the study of the systems with polynomial coefficients with the operator $\frac{d}{dx}$ (usual derivation) for systems with the differential Bessel's operator $S_\mu = \frac{d^2}{dx^2} - \frac{4\mu^2-1}{4x^2}$, $\mu > -1/2$. He obtains uniqueness classes and classes where the solutions of Cauchy's problem exist among the regular functions with compact support in the interval $I =]\,0, \infty\,[$. In 1985 the author (Gonzalez [8]) generalizes the Lee's theorems and the existence and uniqueness of such solutions were found between the functions with exponential growth, being that this growth must be associated with the type of each system which would be considered.

In this paper the operator S_μ will be studied into Hilbert spaces, which will be denoted by H^q_μ, $\mu \in R$, $q \in N$; being that each space H^q_μ is contained in the classical space $L_2(0,\infty)$ S_μ is the infinitesimal generator of a semigroup of bounded $T_\mu(t)$, which will be completely characterized. A convolution defined between measurable functions, which was defined by the author (Gonzalez [7]) will allow us to find an explicit expression of $T_\mu(t)$. Then it will be proved that the semigroup $T_\mu(t)$ is an analytic semigroup of bounded linear opdrators. The previous theorems about S_μ and $T_\mu(t)$ will lead us to the study of Cauchy's problem for an equation of parabolic type which contains the operator S_μ.

Other partial differential equations with different Bessel type operators can be solved likewise with the help of the theorems which will be obtained for the operator S_μ. Certain of these equations will be studied in the last paragraph of the paper, since they appear as significant cases of the Fokker–Planck equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left[A(x)\ \frac{\partial}{\partial x}[\,u(x,t)\,]\right] + \frac{\partial}{\partial x}\left[B(x)\ u(x,t)\right] \quad \ldots (1.1)$$

for $x \in I$, $t \in 0,T$, and $A(x)$ and $B(x)$ being coefficients which are always known.

2. SPACES H_μ^q AND L^q, $q \in N$

We denote by X the Hilbert space $L_2(0,\infty)$ with the inner product:

$$(\phi/\Phi) = \int_0^\infty \phi(x)\,\Phi(x)\,dx \ \forall\ \phi,\Phi \in X \quad \ldots (2.1)$$

The classical Hankel transformation h_μ, which is defined by the pair (2.2):

$$\begin{aligned}(h_\mu\phi)(\sigma) &= \int_0^\infty (x\sigma)^{1/2} J_\mu(x\sigma)\,\Phi(x)\,dx \\ (h_\mu^{-1}\Phi)(x) &= \int_0^\infty (x\sigma)^{1/2} J_\mu(x)\,\Phi(\sigma)\,d\sigma;\ x,\sigma \in I\end{aligned} \quad \ldots (2.2)$$

verifies in X the following theorem 1 :

Theorem 1. (Bochner [1]) The linear operator h_μ establishes an automorphism on X.

We must understand the formulation of the previous theorem as follows:

(a) Each function $F_n(\sigma) = \int_0^n (x\sigma)^{1/2} J_\mu(x\sigma), f(x), \forall\ n \in N$ belongs to the space X, being $f \in X$. Moreover the sequence $\{F_n(\sigma)\}_{n \in N}$ converges in the norm of X, being $\lim_{n\to\infty} F_n(\sigma) = \int_0^\infty (x\sigma)^{1/2} J_\mu(x\sigma)\,f(x)dx$

(b) $\int_0^\infty |(h_\mu f)(\sigma)|^2\,d\sigma = \int_0^\infty |f(x)|^2\,dx$

(c) $f(x) = h_\mu\left[\ (h_\mu f)(\sigma)\ \right](x),\ \forall\ f \in X$

For each real number μ and each integer q, we define a Hilbert space H^q_μ as follows. A function $\phi(x)$ is in H^q_μ if and only if it is defined on the interval I, it is real-valued and smooth and, for each nonnegative integer $r < q$ the expression:

$[N_{\mu+r-1}.\ N_{\mu+r-2} \dots N_\mu(\phi)](x)$ belongs to X; being N_μ the differential operator:

$$(N_\mu\phi)(x) = x^{\mu+1/2}\ \frac{d}{dx}\left[x^{-\mu-1/2}\ \phi(x)\right],\ x>0$$

H^q_μ is a liner space. Also we can define the norm (2.3):

$$\|\phi\|^q_\mu = \sum_{r=1}^{q}\left[\int_0^\infty |(N_{\mu+r-1}.\ N_{\mu+r-2} \dots (N_\mu)(\phi(x))|^2\,dx\right]^{1/2} \quad (2.3)$$

Thus, we can prove the following lemma:

Lemma 1. The space H^q_μ is dense in X, for any $q \in N$ and $\mu>0$.

Proof: The lemma will be proved on seeing that the Fréchet space H_μ which was introduced by Zemanian in the form:

$H_\mu = \{\phi: I \to C\ /\phi$ is a smooth function, being that the expressions:

$$\sup_{x\in I}\left|x^m\ (x^{-1}\frac{d}{dx}\left[x^{-\mu-1/2}\ \phi(x)\right]\right| \text{ remain bounded }\},$$

it is contained into the space H^q_μ $\forall\ q \in N$ because the composition of differential operators:

$[N_{\mu-r-1}.\ N_{\mu+r-2} \dots N_\mu\ \phi\ (x)]$

coincides with the expression:

$$x^{\mu+r-1/2}\left[x^{-1}\ \frac{d}{dx}\right]^r\left[x^{-\mu-1/2}\ \phi\ (x)\right],\ \forall\ \phi \in H_\mu$$

which appears inside the definition of H_μ.

Furthermore, the space of smooth functions with compact $D(I)$ is dense in X as:

$$D(I) \subset H_\mu \subset H_\mu^q \subset X$$

we will have proved that H_μ^q is dense in X.

We must now define the space L^q for $q \in N$. A function $\phi: I \to C$ belongs to the space L^q if and only if, for any nonnegative integer $r: r \leqslant q$, the fpnction $\sigma^r \phi(\sigma)$ belongs to X.

L^q is a normed space when the norm (2.4) would be defined:

$$\Gamma^q(\phi) = \sum_{k=1}^{q} \left[\int_0^\infty | \sigma^r \phi(\sigma) |^2 d\sigma \right]^{1/2} \qquad \dots (2.4)$$

Then, we can prove the following theorem:

Theorem 2. The linear operator h_μ: $H_\mu^q \to L^i$; h_μ: $\phi \to h_\mu(\phi)$ defines an isometric isomorphism from the space H_μ^q onto the space L^q, for $\mu > -1/2$

Proof: (a) Theorem 1 proves this theorem when $q = 0$

(b) Being $q > 0$, $q \in N$ and $\phi \in H_\mu^q$ the formulae:

$$\sigma^r \left[h_{\mu'} \phi) \right](\sigma) = (-1)^r h_{\mu+r} \left[(N_{\mu+r-1} \cdot N_{\mu+r-2} \dots N_\mu)(\phi) \right] \sigma$$

easily can be proved for any integer $r \leqslant q$ and $\sigma \in I$ and for when the recurrence relations of the Bessel function $J_\mu(z)$ are known. Then. we can use Theorem 1, and in this way we will find that $h_\mu \phi$ belongs to L^q, h_μ being an isometry.

(c) When we choose a function Φ belonging to the space L^q, $q > 0$, we know that:

$$\left[N_{\mu+r-1}\cdot N_{\mu+r-2}\cdots N_{\mu}\right]\left[h_{\mu}\Phi\right](\sigma)=(-1)^{r}\,h_{\mu+r}\left[\sigma^{r}\,\Phi(\sigma)\right](x)$$

for any integer $r \leqslant q$, *i. e.*, the function $\left[N_{\mu+r-1}\cdot N_{\mu+r-2}\cdots N_{\mu}\right]\left[h_{\mu}\Phi\right](\sigma)$ belongs to the space X, and in consequence the function $(h_{\mu}\Phi)(x)$ belongs to the space H_{μ}^{q}.

Therefore, we also have that the inverse mapping h_{μ}^{-1} coincides with h_{μ}.

3. SEMIGROUP OF BOUNDED OPERATORS $T_{\mu}(t)$

In the Hilbert space X we define the operator A associated with the differential operator S_{μ} as follows:

A is the operator:

$A: D(A) \subset X \to X$

$$\phi \to A(\phi)(x) = S_{\mu}\phi(x) = x^{-\mu-1/2}\,\frac{d}{dx}\,x^{2\mu+1}\frac{d}{dx}\,x^{-\mu-1/2}\,\phi(x) \quad \dots (3.1)$$

$D(A)$ being the domain of A, given in

$$D(A) = \left\{ \phi \in X / \text{ the function } S_{\mu}\phi\,(x) = \frac{d^{2}\phi}{dx^{2}} - \frac{4\mu^{2}-1}{4x^{2}}\,\phi(x) \text{ is defined, being that } S_{\mu}\phi \text{ belongs to the space } X \right\}.$$

A characterization of the domain $D(A)$ was obtained by Callias (3) in the from: When $(A'\phi)(x)$ denotes the operator: $(A'\phi)(x) = -\dfrac{d^{2}\phi(x)}{dx^{2}} + \dfrac{K}{x^{2}} - \phi(x)$, K being a real number: $K \geqslant 3/4$ if a function $\phi \in X$ be in the domain of the closure of the operator A' defined on $C_{0}^{\infty}(0,\infty)$; extending ϕ to R_{-} by $\phi=0$ in R_{-}, then we have: $\phi \in H^{1}(R) = \{\phi \in L_{2}(R) / (1+\sigma^{2})^{1/2}\,\bar{\phi}(\sigma) \in L_{2}(R)\}$, ϕ' is absolutely continuous on I and $\lim_{x\to 0} \phi(x) = 0$ ($\tilde{\phi}$ is the Fourier transform of ϕ).

The space H_μ^{q} is contained in $D(A)$ for $q>2$, since:

$$(S_\mu\phi)(x) = x^{-\mu-1/2}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx}x^{-\mu-1/2}\phi(x))=\Big[2(\mu+1)N_\mu+N_{\mu+1}.\,N_\mu\Big](\phi(x)) \quad \ldots (3.2)$$

then, as a consequence of (3.2), we can prove the following properties:

(a) $D(A)$ is dense in X

(b) $-A$ is a monotone operator, because:

$$(-A\phi/\phi = -\int_0^\infty (x^{-\mu-1/2}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx}\Big[x^{-\mu-1/2}(\phi(x))\Big]\phi(x)\,dx =$$

$$= \int_0^\infty |N_\mu\phi|^2\,dx \geqslant 0 \text{ for } \phi\in D(A) \quad \ldots (3.3)$$

(c) A is closed operator

We want to see which is the resolvent set $\rho(A)$ of the operator A, i. e.:

$$\rho(A) = \{(\lambda\in C/\ \lambda I - A) \text{ is a bounded linear in } X\}$$

Then, let f be a function of X. We must find a function g in $D(A)$ being that:

$$(\lambda I - A)(g) = f \quad \ldots (3.4)$$

Firstly, we consider that f belongs to the space $D(I)$, and then, applying h_μ to the two members of (3.4) we will obtain:

$$(h_\mu g)(\sigma) = \frac{1}{\sigma^2+\lambda}(h_\mu f)(\sigma)$$

Thus, the function $g(x)$ can be expressed in the form:

$$g(x) = h_\mu\Big[\frac{1}{\sigma^2+\lambda}(h_\mu f)(\sigma)\Big](\sigma\to x) \quad \ldots (3.5)$$

We can simplify the expression (3.5) of g with the help of a convolution product which was studied by the author (González [7]). Its definition is given in the following:

Given f and g two measurable functions in I, the convolution $f*g$ between f and g is defined by:

$$f*g)(x) = x^{\mu+1/2} \int_0^\infty \overset{\nu}{\underset{x}{T}} \left[u^{-\mu-1/2} g(u) \right] f(y)\, y^{\mu+1/2}\, dy, \quad x > 0 \qquad \text{... (3.6)}$$

where $\overset{\nu}{\underset{x}{T}}$ denotes the translation operator:

$$\overset{\nu}{\underset{x}{T}} g = \frac{1}{\sqrt{\pi}\, 2^{\mu} \Gamma(\mu+1/2)} \int_0^\pi g([x^2+y^2-2xy\cos\omega]^{1/2}) \sin^{2\mu}\omega\, d\omega \quad \text{...(3.7)}$$

We can prove

Lemma 2. When the functions f and g verify:

$y^{\mu+1/2} f(y)$, $y^{\mu+1/2} g(y) \in L_1(0,\infty)$, then the convolution $f*g$ belongs to $L_1(0,\infty)$, too, moreover:

$$h_\mu(f*g)(\sigma) = \sigma^{-\mu-1/2} (h_\mu f)(\sigma).\ (h_\mu g)(\sigma), \quad \sigma \in I \qquad \text{... (3.8)}$$

As a consequence of Lemma 2, we find that $g(x)$ in (3.5) admits the representation [14, p. 108]:

$$g(x) = h_\mu \left[\sigma^{-\mu-1/2} h_\mu \left[\lambda^{\mu/2} y K_\mu(\lambda y) \right] (h_\mu f)(\sigma) \right](x)$$

where K_μ is the modified Bessel function of the third class and order μ. And then:

$$g(x) = \left[\lambda^{\mu/2} f(y) * y K_\mu(\lambda y) \right](x)$$

Thus [16, p. 367]:

$$\overset{\nu}{\underset{x}{T}} \left[u^{-\mu-1/2}\, u\, K_\mu(\lambda u) \right] = \lambda^{\mu/2} \lambda^{-\mu/2} x^{-\mu/2} \lambda^{-\mu/2} y^{-\mu/2} K_\mu(\lambda x) J_\mu(\lambda y) \quad (3.9)$$

Then, the function $g(x)$ which is the solution of the resolvent equation will be:

$$g(x) = \sqrt{x}\, K_\mu\, (\sqrt{\lambda} x) \int_0^\infty f(y) \sqrt{y}\, J_\mu\, (\sqrt{\lambda} y)\, dy =$$

$$= \sqrt{x}\, K_\mu\, (\sqrt{\lambda} x)\, \lambda^{-1/4} h_\mu\, [f(y)]\, (\lambda) \qquad \ldots (3.10)$$

Moreover, the composition of the operators $N_{\mu+1}.$ and N_μ give us:

$$N_{\mu+1}.\, N_\mu\, (g(x)\,) = N_{\mu+1}.\, N_\mu\, R(\lambda : A)\,(f) = \lambda\, R\, \lambda : A)\,(f)$$

such that, then, we can prove that the searched function $R\,(\lambda : A)(f)$ belongs to the domain $D(A)$ since the norm:

$$\| R(\lambda : A)(f) \|_\mu^2 \leqslant \int_0^\infty |\sqrt{x}\, K_{\mu-2}\, (\sqrt{\lambda} x)\,|^2 \left[\left[\int_0^\infty |f(y)|^2\, dy\right.\right.$$

$$\left[\left[\int_0^\infty |\, J_\mu\, (\sqrt{\lambda} t) \sqrt{t}\,|^2\, dt\right]\right] dx$$

remains bounded:

Until now, the function f had been chosen between the regular functions with bounded support in I. When f belongs to the space X the previous results can be proved too; and, in addition it is possible to obtain a constant $M > 0$, such that the norm of $R(\lambda : A)$ verifies:

$$\| R(\lambda : A) \| \leqslant M/\lambda, \quad \forall\ \lambda \in \rho(A) \qquad \ldots (3.11)$$

Then we can prove

Theorem 4. The differential operator S_μ is the infinitesimal generator of a C_0 semigroup $T_\mu(t)$.

Following the proof of Theorem 4 we know (Pazy [15]) that the semigroup $T_\mu(t)$ can admit the following representation:

$$T_\mu(t)(g)\,(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} \Big[R(\lambda : A) g \Big]\,(x)\, d\lambda =$$

$$= \int_0^\infty g(y) \sqrt{y}\, (-i)^\mu\, L^{-1} \Big[\, K_\mu\, (\sqrt{\lambda} x)\, I_\mu\, (-\sqrt{\lambda} y i) \,\Big](t)\, dy$$

where g belongs to X and L^{-1} denotes the inverse Laplace transformation. However, the inverse transformation of $K_\mu(\sqrt{\lambda}x)I_\mu(-\sqrt{\lambda}\,yi)$ is known

(Ditkin-Proudnikov [4,p. 344], so we can obtain the expression of $T_\mu(t)$ given in (3.12):

$$[T_\mu(t)g](x) = x(2t)^{-1}\Big[e^{-x^2/4t} \int_0^\infty g(y)\sqrt{y}\,J_\mu(xy/2t)\, e^{y^2/4t}\, dy +$$

$$+\; e^{-x^2/4t}\int_0^\infty g(y)\sqrt{y}\,J_\mu(xy/2t)\, e^{-y^2/4t}\, dy\Big] \qquad \dots (3.12)$$

Moreover, if we say that the semigroup $T_\mu(t)$ is analytic when there exist two nonnegative constants $\triangle$ and C and

$$\| A\, R(\lambda:A)^{n+1} \| \leqslant C/n\lambda^n,\ \forall\ n \in N \text{ and } \forall\ \lambda \geqslant n\triangle \qquad \dots (3.13)$$

then

$$[R(\lambda:A)^2 f](x = \sqrt{x}K_\mu(\lambda x)\int_0^\infty J_\mu(\sqrt{\lambda}y)\sqrt{y}\,f(y)dy$$

$$\Big[\int_0^\infty \sqrt{u}K_\mu(\sqrt{\lambda}u\, J_\mu(\sqrt{\lambda}u)\ du\Big] = (2\lambda)^{-1}\,[R(\lambda:A)f]\,(x)$$

it will be:

$$[R(\lambda:A)^{n+1}f]\,(x) = (2\lambda)^{-n}\,[R(\lambda:A)f]\,(x)\ \ \forall\ n \in N \qquad \dots (3.14)$$

and:

$$[AR(\lambda:A)^{n+1}f]\,(x) = 2^{-n}\,\lambda^{1-n}\,[R(\lambda:A)f]\,(x)$$

being, then that the following theorem has been proved:

Theorem 5. The operator S_μ is the infinitesimal generator of an analytic semigroup $T_\mu(t)$, given as the formula (3.12)

4. CAUCHY'S PROBLEM FOR A PARTIAL DIFFERENTIAL EQUATION OF PARABOLIC TYPE WITH THE OPERATOR S_μ.

let us consider the following partial differential equation:

$$\frac{\partial u}{\partial t}(x,t) = \left[\frac{d^2}{dx^2} - \frac{4\mu^2-1}{4x^2}\right](u(x,t)),\ x \in I,\ t \geqslant 0 \qquad \dots (4.1)$$

with the intial values condition:

$$u(x,0) = u_0(x) \qquad \dots (4.2)$$

When the solution $u(x,t)$ of (4.1) and (4.2) would be understood as a abstract function:

$$u :]\,0.\infty\,[\to X$$

$$t \to u(t) : I \to R$$

$$x \to u(t)(x) = u(x,t)$$

the semigroups of linear operators theory can be used in the resolntion of the partial diferential equations and then we can prove the theorem 6.

Theorem 6. When $u_0(x)$ is a function which belongs to the space X, then there is an unique solution $u(x,t)$ of (4.1) and (4.2) in the space X.

Moreover, if Problems (4.1) and (4.2) are inhomogeneous in the form:

$$\frac{\partial u}{\partial t}(x,t) = S_\mu[u(x,t)] + f(x,t) \qquad \dots (4.3)$$

$$u(x,0) = u_0(x),\ x \in I,\ 0 \leqslant t \leqslant T < \infty \qquad \dots (4.4)$$

we will have

Theorem 7. If the function f belongs to the space $L_1(0,T,X)$ and it is locally Holder continuous in $[0,T]$, for each $u_0(x)$ belonging to X, there is a unique solution $u(x,t)$ of (4.3) and (4.4) which admits the expression:

$$u(x\ t) = T_{\mu}(t)\ [u_0\,(x)] + \int_0^t T_{\mu}'t\text{-}s(\ [f(x,s)]\ ds$$

Theorems 6 and 7 can be modified and then they may be used in the study of other parabolic equations of type (4.1)-(4.2) where these appear different Bessel-type differential operators of S_{μ}. These operators which present a general expression in the form $x^2\frac{d}{dx}\,x^2\,\frac{d}{dx}\,x^c; a+b+c<2$ have been studied by a group of research workers of the Mathematical Analysis Department of the University of La Laguna (Canaary Islands). They have found every one of the topics relative to the operational treatment of those operators; i.e. they have found integral transformations, spaces of test functions where these operators define linear and continuous mappings; convolution product associated with them; etc. In particular, the Bessel-Clifford's operator $B_{\mu} = x.\,\frac{d^2}{d\lambda^2} + (1-\mu)\,\frac{d}{dx}$ $= \frac{d}{dx}\,x^{\mu+1}\,\frac{d}{dx}\,x^{-\mu}$, $\mu > 0$ has been carefully analysed (González, [7]).

We can define

$$\frac{\partial u}{\partial t}\,(x,t) = B_{\mu}\,[u(x,t)] \quad x > 0 \quad t \geqslant 0 \qquad \ldots (4.5)$$

being the initial value problems (4 5) and (4 6):

$$u(x,0) = u_0(x) \qquad \ldots (4\ 6)$$

could be studied with the help of the previous theorems.

So, when $u(x,t)$ is a function which belongs ($\forall\ t \geqslant 0$) to the space

$$X' = \left\{ f.\ I \rightarrow R \,/\, \int_0^{\infty} |f(u)|^2\,u^{-\mu}\ du\ < \infty \right\}$$

which is a normed space with the norm:

$$\|f\|_{\mu,2} = \left[\int_0^{\infty} |f(u)|^2\,u^{-\mu}\,du \right]^{1/2} f \in X'$$

then the function:

$$v(x,t) = 2^{-1}x^{1/2-\mu}\ u(x^2/2,t)$$

will belong to the space X in 2, $\forall\ t \geqslant 0$. Moreover, we can formally find the relation:

$$S_\mu[x^{-\mu+1/2}\, u(x^2/2,t)] = x^{-\mu+1/2}\, B_\mu\, [u(x,t)]\, (x^2/2),\ \forall\ x \in I,\ \ t > 0$$

and then, when $v(x,t)$ could be a solution of (4.1) - (4.2) defined into the space X, $u_0(x)$ belonxing to the space X; $u(x,t)$ will be a solution in X' of (4.5) and (4.6) for $v_0(x)$ chosen in the space X'. In this way, the initial value problem (4.5) and (4.6) can be solved, and then two theorems like the 6 and 7 ones are thus proven.

The previous operational treatment of parabolic equations can also be used likewise in the study of equations of the same type with other Bessel type differential operators; among others the following ones:

$$x^{-\mu}\frac{d}{dx}\, x^{\mu+1}\frac{d}{dx} = x\,\frac{d^2}{dx^2} + (\mu+1)\,\frac{d}{dx} = \frac{d}{dx}\left[\, x\,\frac{d}{dx}\right] + \mu\frac{d}{dx}$$

or

$$x^{-2\mu-1}\frac{d}{dx}\, x^{2\mu+1}\frac{d}{dx} = \frac{d^2}{dx^2} + \frac{2\mu+1}{x}\,\frac{d}{dx}\,.$$

These differential operators usually appear in particular cases of Fokker Planck equation (1.1); precisely in those cases where the said equation has the most outstanding biological and physical significance (González, [9]).

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SOME INEQUALITIES OF HYPERGEOMETRIC FUNCTION OF THREE VARIABLES

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ABSTRACT

The set of hypergeometric functions of three variables for which the inequalities are known is extended to include the inequalities of the Srivastava functions H_A, H_B and H_C.

1. INTRODUCTION

Lukes technique [9] of obtaining inequalities through Euler integral representations (which he used in obtaining inequalities of the Gauss and Appel's hypergeometric functions) was used by Joshi and Arya [7] to unfold the inequalities of some of the functions of the Lauricella's set. viz. F_A, F_B, F_D, F_G, F_M, F_N, F_K, F_S and F_T. The other known inequalities of functions of three variables are due to Deshpande [4] for F_K and Bhise and Dange [2] for F_M (see also Srivastava and Karlsson [16, pp. 344-346]).

In this paper we enlarge the existing list of inequalities of three variables by obtaining the inequalities of H_A. H_B and H_C. It would be of interest to recall that H_A, H_B, H_C were discovered by Srivastava [14] in the course of his attempt to generalize Lauricella's 14 hypergeometrtc series of three variables when he noticed the existence of three additonal complete hypergeometric serties of second order. In Sections 2 and 3 inequalities for H_A and H_C are obtained through Euler integral representation technique, while in Section 4 inequalities for H_B and also those of H_A and H_C are obtained through Laplace type integral representation and ${}_0F_1$ inequalites for positive and negative real arguments.

2. INEQUALITY FOR H_A

The function [[16], p. 43 (11)]

$$H_A = H_A(\alpha,\beta,\beta' ; \gamma, \gamma' ; -x, -y, -z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}\,(\beta)_{m+n}\,(\beta')_{n+p}}{(\gamma)_m\,(\gamma')_{n+p}}\;\frac{(-x)^m}{m!}\;\frac{(-y)^n}{n!}\;\frac{(-z)^p}{p!},$$

$|x| > r', |y| < s', |z| < t', r' < 1, t' < (1-r')(1-s'),$

none of γ or γ' is a negative integer or zero, can be expressed as

$$H_A = \frac{\Gamma(\gamma')}{\Gamma(\beta')\,\Gamma(\gamma'-\beta')} \int_0^1 u^{\beta'-1}(1-u)^{\gamma'-\beta'-1}\,(1+uy)^{-\beta}\,(1+uz)^{-\alpha}$$

$$\cdot\ {}_2F_1\left(\alpha, \beta; \gamma; \frac{-x}{(1+uy)(1+uz)}\right) du, \quad R(\gamma') > R(\beta') > 0. \qquad \ldots (2.1)$$

An application of Luke's inequality ([9], 4.7) to the inner ${}_2F_1$ in (2.1) would give

$$\frac{\Gamma(\gamma')}{\Gamma(\beta')\,\Gamma(\gamma'-\beta')} \int_0^1 u^{\beta'-1}\,(1-u)^{\gamma'-\beta'-1}\,(1+uy)^{-\beta}\,(1+uz)^{-\alpha}.$$

$$\cdot\ \left(1 + \frac{\alpha\,\beta\,x}{\gamma(1+uy)\,(1+uz)}\right)^{-1} du$$

$$< H_A < \frac{\Gamma(\gamma')}{\Gamma(\beta')\,\Gamma(\gamma'-\beta')} \int_0^1 u^{\beta'-1}\,(1-u)^{\gamma'-\beta'-1}\,(1+uy)^{-\beta}\,(1+uz)^{-\alpha}.$$

$$\left[1 - \frac{2\alpha\beta(\gamma+1)}{\gamma(\alpha+1)\,(\beta+1)} + \frac{2\alpha\beta(\gamma+1)}{\gamma(1+\alpha)\,(1+\beta)}\left\{1 + \frac{(\alpha+1)\,(\beta+1)x}{2(\gamma+1)(1+uy)(1+uz)}\right\}^{-1}\right] du,$$

$$\gamma > \beta > 0,\ \gamma' \geqslant \beta'\ 0,\ 0 < \alpha \leqslant 1,\ t' < (1-r')(1-s'),\ x > 0. \quad \ldots (2.2)$$

Now if we write

$$1 + au + bu^2 = (1+us)\,(1+ur),$$

so that

$$s = \frac{a+(a^2-4b)^{1/2}}{2}, r = \frac{a-(a^2-4b)^{1/2}}{2}, \quad \dots (2.3)$$

where $a \rhd 2\sqrt{b}$,

further approximation of $(1+uy)^{-\beta}$ and $(1+uz)^{-\alpha}$ by (4.4) of [9], using (2.3), appropriate partial fractions and (4.6) of [9],

we have

Theorem 1: If $0 \lhd \alpha \leqslant 1$, $0 \lhd \beta \leqslant 1$, $\gamma > \beta > 0$, $\gamma' > \beta' \rhd 0$, $y > s > r \rhd z \rhd \beta y > \alpha z > 0$, $y > s_1 > r_1 > z > 0$ aad $yz > r_1 s_1 > 0$,

then

$$a_0 \left[\frac{r(y-r)(r-z)}{a_3\, a_4\, a_5}\, e_3 + \frac{\beta y(z-\beta y) c_4}{a_1\, a_4\, a_6}\, e_4 - \frac{s(y-s)(s-z)}{a_1\, a_2\, a_3}\, (b_2+b_3 e_1) \right.$$

$$\left. - \frac{\alpha z(y-\alpha z) c_3}{u_2\, a_5\, a_6} (b_2+b_3 e_3). \right] < H_A$$

$$< (1-b_8) \left[\frac{c_3\, c_4}{c_1\, c_2} + (b_2+b_3 e_5 (b_6 + \frac{b_7 c_2 y}{c_6} + b_5(b_2+b_3) - \frac{b_7 c_1 z}{c_6}\, e_7 \right]$$

$$+ b_0 \left[\frac{c_3 c_4}{c_1 c_2} \left(\frac{yz}{r_1 s_1} + \frac{r_1 s_1 (y+z) - yz(r_1+s_1)}{r_1\, s_1\, d_1} (s_1(b_2+b_3 e_8) - r_1 e_9) \right. \right.$$

$$\left. + \frac{(yz-r_1 s_1)}{r_1\, s_1\, d_1} (r_1(b_2+b_3 e_0) - s_1 e) \right)$$

$$+ b_6 \left(\frac{y c_4 c_5}{c_9 d_7} (b_2+b_3 e_5) - \frac{2 c_7 c_8}{c_9 d_1}\, e_9 - \frac{2 d_2 d_3}{d_1 d_7}\, e \right)$$

$$+ b_5 \left(\frac{z c_3 d_4}{d_5 d_6} (b_2+b_3 e_6) + \frac{2 c_7 c_8}{d_1 d_5} (b_2+b_3 e_0) - \frac{2 d_2 d_3}{d_1 d_6} e \right)$$

$$\left. + b_7 \left(\frac{c_2 c_4 c_5 y^2}{c_6 c_9 d_7} (b_2+b_3 e_5) - \frac{4 s_1 d_2 d_3}{d_1 d_6 d_7}\, e' - \frac{4 r_1 c_7 c_8}{c_9 d_1 d_6}\, e_9 - \frac{c_1 c_3 d_4 z^2}{c_6 d_5 d_6} e_7 \right) \right]$$

where

$$s = \frac{1}{2}(a + \sqrt{(a^2-4b)}),\ r = \frac{1}{2}(a-\sqrt{(a^2-4b)}),\ a = (y+z)a_0,$$

$$b = yza_0,\ a_0 = (1 + \frac{\alpha\beta x}{\gamma})^{-1},$$

$$s_1 = \frac{1}{2}(c + \sqrt{(c^2-4d)}),\ r_1 = \frac{1}{2}(c-\sqrt{(c^2-4d)}),\ c = (y+z)b_0\ ,$$

$$d = yzb_0,\ b_0 = \left(1 + \frac{c_1c_2x}{2(1+\gamma)}\right)^{-1},$$

$$a_1=s-\beta y,\ a_2=s-\alpha z,\ a_3=s\ r,\ a_4=r-\beta y,\ a_5=r-\alpha z,\ a_6=\beta y-\alpha z,$$

$$b_1 = \frac{1+\beta'}{1+\gamma'},\ b_2 = \frac{\gamma'-\beta'}{\gamma'(1+\beta')},\ b_3 = \frac{\beta'(1+\gamma')}{\gamma'(1+\beta')},\ b_4 = \frac{\beta'}{\gamma'},\ b_5 = \frac{2\alpha c_4}{c_1c_2},$$

$$b_6 = \frac{2\beta c_3}{c_1c_2},\ b_7 = \frac{4\alpha\beta}{c_1c_2},\ b_8 = \frac{2\alpha\beta(1+\gamma)}{\gamma c_1c_2},$$

$$c_1 = 1+\alpha,\ c_2 = 1+\beta,\ c_3 = 1-\alpha,\ c_4 = 1-\beta,\ c_5 = c_2y-2z,$$

$$c_6 = c_2y-c_1z,\ c_7 = y-r_1,\ c_8 = r_1-z,\ a_9 = c_2y-2r_1,$$

$$d_1 = s_1-r_1,\ d_2 = y-s_1,\ d_3 = s_1-z,\ d_4 = 2y-c_1z,\ d_5 = 2r_1-c_1z,$$

$$d_6 = 2s_1-c_1z\quad d_7 = 2s_1-c_2y,$$

$$e' = (1 + \frac{b_4c_2y}{2})^{-1},\ e_0 = (1+b_1r_1)^{-1},\ e = (1+b_4s_1)^{-1},\ e_1 = (1+b_1s)^{-1}$$

$$e_2 = (1+b_4r)^{-1},\ e_3 = (1+b_1\alpha z)^{-1},\ e_4 = (1+\beta b_4y)^{-1},\ e_5=(1+b_1c_2y/2)^{-1},$$

$$e_6 = (1+b_1c_1z/2)^{-1},\ e_7=(1+b_4c_1z/2)^{-1},\ e_8=(1+b_1s_1)^{-1},\ e_9=(1+b_4r_1)^{-1}.$$

3. INEQUALITIES H_G

The functions is defined by ([16], p. 43 (13))

$$H_C = H_C\ (\alpha,\ \beta,\ \beta';\ \gamma;\ -x,\ -y,\ -z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}\,(\beta)_{m+n}\,(\beta')_{n+p}}{(\gamma)_{m+n+p}\; m!\,n!\,p!} (-x)^m(-y)^n(-z)^p,$$

γ is not a negative integer or zero and $|x| < r'$, $|y| < s'$, $|z| < t'$ such that $r' < 1$, $s' < 1$ and $r'+s'+t'-2\sqrt{|(1-r')(1-s')(1-t')|} < 2$. ... (3.1)

Rewriting H_C as

$$H_C = \sum_{m,n=0}^{\infty} \frac{(a)_m\,(\beta')_n\,(\beta)_{m+n}}{(\gamma)_{m+n}}\; {}_2F_1\,(a+m,\ \beta'+n;\ \gamma+m+n;\ -z)\,\frac{(-x)^m(-y)^n}{m!\,n!},$$

and approximating ${}_2F_1$ by ([16], p. 180 (2)), we have

$$(1+z)^{\gamma-\alpha-\beta'}\,F_1\,(\beta,\ a,\ \beta';\ \gamma;\ -x,\ -y) < H_C$$

$$< \frac{\Gamma(\alpha+\beta'-\gamma)\Gamma(\gamma)}{\Gamma(a)\;\Gamma(\beta')}\,(1+z)^{\gamma-\alpha-\beta'} \sum_{m,n=0}^{\infty} \frac{(\beta)_{m+n}\,(x)^m(-y)^n}{m!\;n!},$$

$\gamma > \alpha \geqslant \gamma-\beta' > 0$, $z > 0$. ... (3.2)

Further approximating the F_1 in left-hand side of (3.2) by Luke's result ([10], p. 76 (8)), we have

Theorem 2: If $\gamma > \alpha \geqslant \gamma-\beta' > 0$, $0 < a \leqslant 1$, $0 < \beta' \leqslant 1$, $\beta' \geqslant \alpha > 0$, $\gamma > \beta > 0$, and $y \geqslant x > 0$, then

$$(1+z)^{\gamma-\alpha-\beta'}\left[\frac{\beta' y}{(\beta' y-a x)}\left(1+\frac{\beta\beta' y}{\gamma}\right)^{-1} - \frac{a x}{(\beta' y-a x)}\left(\frac{\gamma-\beta}{\gamma(1+\beta)} + \frac{\beta(1+\gamma)}{\gamma\ 1+\beta)}\,(1+\frac{a x(1+\beta)}{(1+\gamma)})^{-1}\right)\right]$$

$$< H_C < \frac{\Gamma(\alpha+\beta'-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\;\Gamma(\beta')}\,(1+z)^{\gamma-\alpha-\beta'}\,(1+x+y)^{-\beta}. \qquad \text{... (3.3)}$$

However, applying ([6], p. 180 (2)) in (3.2) and then approximating the ${}_2F_1$ by (4.7) of [9], we have

Theorem 3: If $\gamma > \alpha > \gamma-\beta' > 0, \gamma > \beta' > \gamma-\beta > 0, 0 < \alpha \leqslant 1,$
$x > 0, y > 0,$ then

$$(1+z)^{\gamma-\alpha-\beta'} (1+y)^{\gamma-\beta-\beta'}\left(1 + \frac{\alpha\beta x}{\gamma}\right)^{-1}$$

$$< H_C < \frac{\Gamma(\alpha+\beta'-\gamma)\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta')} (1+z)^{\gamma-\alpha-\beta'} (1+x+y)^{-\beta}. \qquad \ldots (3.4)$$

Note also that H_C admits the integral representation

$$H_C = \frac{\Gamma(\gamma)}{\Gamma(\beta')\,\Gamma(\gamma-\beta')} \int_0^1 u^{\beta'-1} (1-u)^{\gamma-\beta'-1} (1+uy)^{-\beta} (1+uz)^{-\alpha}$$

$$\cdot\, {}_2F_1 \left[\begin{matrix}\alpha, \beta\ ; \\ \gamma-\beta'\ ;\end{matrix} \frac{-x\,(1-u)}{(1+uy\ \ (1+uz)}\right] du,$$

$$\gamma > \beta' > 0. \qquad \ldots (3\ 5)$$

Approximating inner ${}_2F_1$ by (4.7) of [9] then appropriately using (2.3). (4.4) of [9], pratial fractions and Luke's result ([9], 4.6) we are led to

Theorem 4. If $0 < \alpha \leqslant 1, 0 < \beta \leqslant 1, \gamma > \beta' > 0,$
$y > s_2 > s_3 > z > \beta y > \alpha z > 0$ and $y > s_3 > s_4 > z > 0,$ then

$$f_0 \left[\frac{\beta y(z-\beta y)c_4}{f_1 f_2 f_3} k_0 - \frac{\alpha z(y-\alpha z)c_3}{f_3 f_4 f_5} (f_7+f_8k_1) - \frac{s_2(y-s_2)(s_2-z)}{f_1 f_4 f_6} (f_7+f_8k_2)\right.$$

$$\left. + \frac{s_3(y-s_3)(s_3-z)}{f_2 f_5 f_6} k_3\right]$$

$$< H_C$$

$$<(1-g_2)\left[\frac{c_3 c_4}{c_1 c_2}+ g_3(f_7+f_8k_8)+g_4(f_7+f_8k_9) + \frac{4\alpha\beta y}{c_1 g_5} (f_7+f_8k_8) + \frac{4\alpha\beta z}{c_2 g_5} k'\right]$$

$$+ h_0 \left[\frac{c_3c_4}{c_1c_2} + \left(\frac{yz}{s_4 s_5} + \frac{s_4 s_5(y+z)-yz(s_4+s_5)}{s_4 s_5 h_3}\right. (s_4(f_7+ f_8k_4)-s_5 k_5)\right.$$

$$+ \frac{(y z-s_4s_5)}{s_4 s_5 h_3} (s_5(f_7+ f_8k_6) -s_4k_7))$$

$$+ g_3 \left(\frac{c_4 g_6 y}{g_7 g_8} (f_7 + f_8 k_8) - \frac{2h_1 h_2}{h_3 g_8} k_5 - \frac{2h_4 h_5}{h_3 g_7} k_7 \right)$$

$$+ g_4 \left(\frac{c_3 h_6 z}{h_7 g_9} (f_7 + f_8 k_9) + \frac{2h_1 h_2}{h_3 g_9} (f_7 + f_8 k_6) - \frac{2h_4 h_5}{h_3 h_7} k_7 \right)$$

$$+ \frac{4\alpha\beta}{c_1 c_2} \left(\frac{c_2 c_4 g_6 y^2}{g_5 g_7 g_8} (f_7 + f_8 k_8) - \frac{4 s_4 h_4 h_5}{h_3 h_7 g_7} k_7 - \frac{4 s_5 h_1 h_2}{h_3 g_8 g_9} k_5 - \frac{c_1 c_3 h_6 z^2}{h_7 g_5 g_9} k' \right) \Bigg],$$

where

$$s_2 = \frac{1}{2} (f + \surd(f^2 \;\, 4g)),\; s_3 = \frac{1}{2} (f - \surd(f^2 - 4g)),\; f = (y + z - \frac{\alpha\beta x}{(\gamma - \beta')})\, f_0.$$

$$g = yz f_0,$$

$$s_4 = \frac{1}{2} (h + \surd(h^2 - 4k)), s_5 = \frac{1}{2} (h - \surd(h^2 - 4k)),\; h = (y + z - \frac{c_1 c_2 x}{2(1 + \gamma - \beta')})\, h_0,$$

$$k = yz h_0,$$

$$f_0 = (1 + \frac{\sigma\beta x}{\gamma - \beta'})^{-1},\; f_1 = s_2 - \beta y,\; f_2 = s_3 - \beta y,\; f_3 = \beta y - \alpha z,\; f_4 = s_2 - \alpha z,$$

$$f_5 = s_3 - \alpha z \;\; f_6 = s_2 - s_3,\; f_7 = \frac{\gamma - \beta'}{\gamma(\alpha + \beta')},\; f_8 = \frac{\beta'(1 + \gamma)}{\gamma(1 + \beta')},\; f_9 = \frac{1 + \beta'}{1 + \gamma},$$

$$g_1 = \frac{\beta'}{\gamma},\; g_2 = \frac{2\alpha(\beta(1 + \gamma - \beta')}{c_1 c_2 (\gamma - \beta')},\; g_3 = \frac{2\beta c_3}{c_1 c_2},\; g_4 = \frac{2\alpha c_4}{c_1 c_2},\; g_5 = c_2 y - c_1 z,$$

$$g_6 = c_2 y - 2z,\; g_7 = 2s_4 - c_2 y,\; g_8 = c_2 y - 2s_5,\; g_9 = 2s_5 - c_1 z,$$

$$h_0 = (1 + \frac{c_1 c_2 x}{2(1 + \gamma - \beta'})^{-1},\; h_1 = y - s_5,\; h_2 = s_5 - z,\; h_3 = s_4 - s_5,\; h_4 = y - s_4$$

$$h_5 = s_4 - z,\; h_6 = 2y - c_1 z,\; h_7 = 2s_4 - c_1 z,$$

$$k' = (1 + \frac{c_1 g_1 z}{2})^{-1},\; k_0 = (1 + g_1 \beta y)^{-1},\; k_1 = (1 + f_9 \alpha z)^{-1},\; k_2 = (1 + f_9 s_2)^{-1},$$

$$k_3 = (1 + g_1 s_3)^{-1},\; k_4 = (1 + f_9 s_4)^{-1},\; k_5 = (1 + g_1 s_5)^{-1},\; k_6 = (1 + f_9 s_5)^{-1},$$

$$k_7 = (1 + g_1 s_4)^{-1},\; k_8 = (1 + c_2 f_9 y/2)^{-1},\; k_9 = (1 + c_1 f_9 z/2)^{-1},$$

In particular, for the set of values $\alpha = .6$, $\beta = .7$, $\beta' = .9$, $\gamma = 1$, $x = .2$, $y = .5$, $z = .4$, Theorem 2 gives

$$.6235645 < H_C < .6492608,$$

whereas Theorem 3 gives

$$.6112965 < H_C < .6492608.$$

4. INEQUALITIES FOR H_B AND H_A, H_C

In Sections 2 and 3 inequalities for H_A and H_C are obtained through Euler integral representation but this technique fails to give inequalities for H_B when applied to its known integral representation ([14], p. 100 (3.4)

$$\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma_1-\beta)\Gamma(1-\gamma_1+\beta)\,\Gamma(\gamma_1+\gamma_2-\beta+1)\,\Gamma(\gamma_3-\beta')}{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}\,H_B$$

$$= \int_0^1\int_0^1\int_0^1 u^{\beta-1}\, v^{\beta-\gamma_1}\,(1-u)^{\gamma_1-\gamma_3-\beta+\beta'}\,(1-v)^{\gamma_2+\gamma_3-\beta'-2}\,(1-ux-vy)^{-\alpha}$$

$$.\,(1-u-w+uw-uvwy)^{\gamma_3-\beta'-1}\,du\,dv\,dw,$$

where

$$0 < Re(\gamma_1-\beta) < 1,\ Re(\beta) > 0,\ Re(\gamma_1+\gamma_2-\beta) > 1,\ Re(\gamma_3) > Re(\beta') > 0,$$

and y is chosen sufficiently small, which is found to be erroneous decause $Re(\gamma_1) \not> Re(\beta+n)$, $Re(\gamma_2) \not> (\beta'+n)$, for $n \in N_0$.

It naturally led us to examine if a Laplace-type integral would work. This becomes possible by considering the inequalities

$$1-\frac{z}{\alpha} < {}_0F_1(-;\alpha;-z) < 1+\frac{z}{\alpha}+\frac{z^2}{2(\alpha)_2},\ \alpha > 0,\ z > 0, \qquad \ldots (4.1)$$

and

$$1+\frac{z}{\alpha} < {}_0F_1(-;\alpha;z) < 1+\frac{2z}{\alpha},\ \alpha > 0,\ 0 < z < 1, \qquad \ldots (4.)$$

which are direct consequences of the integral representation

$$ {}_1F_1\,(a;\,c;\,-x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{-x} x^{a-1}\,(1+x)^{c-a-1}\,dx,\quad c > a > 0, $$

the inequalities

$$ 1-z < e^{-z} < 1-z+\frac{z^2}{2},\ z > 0, $$

$$ 1+z < e^{z} < 1+2z,\ 0 < z < 1, $$

and the confluence principle.

Consider the integral representation ([5], p. 75 (3.1.1.2)

$$ \Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\beta')\,H_B\,(\alpha,\,\beta,\,\beta';\,\gamma_1,\,\gamma_2,\,\gamma_3;\,x,\,y,\,z) $$

$$ = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(r+s+t)}\, r^{\alpha-1}\, s^{\beta-1}\, t^{\beta'-1}\, {}_0F_1(-;\,\gamma_1;\,xrs). $$

$$ \cdot\ {}_0F_1\,(-;\,\gamma_2;\,yst)\ {}_0F_1\,(-;\,\gamma_3;\,zrt)\ dr\ ds\ dt, $$

where $R(\alpha)$, $R(\beta)$, $R(\beta')$ are positive and none of $\gamma_1, \gamma_2, \gamma_3$ is a negative integer or zero and $|x| < r'$, $|y| < s'$, $|z| < t'$,
$r' + s' + t' + 2\sqrt{(r's't')} < 1.$ ... (4.3)

Now appropriate application of (4.1) and (4.2) would yield respectively:

Theorem: 5: If $a, \beta, \beta' > 0$ $\gamma_1, \gamma_2, \gamma_3 \geq 0$, and $x, y, z > 0$, then

$$ 1 - \sum_{i=1}^{3} a^i \left[1 - a_{i+1} \left[\frac{1+\alpha_{1+1}}{a_{i+1}} \right] \right] - \frac{xyz(a)_2(\beta)_2(\beta')_2}{\gamma_1\,\gamma_2\,\gamma_3} $$

$$ < H_B\,(a,\,\beta,\,\beta';\,\gamma_1,\,\gamma_2,\,\gamma_3;\,-x,\,-y,\,-z) $$

$$ < 1 - \sum_{i=1}^{3} a_i \left[1 - a_{i+1} \left(\frac{1+a_{i+1}}{\alpha_{i+1}} \right) \right] - \prod_{i=0}^{3} c_i\,(\alpha_i)_2 + \sum_{i=1}^{3} b_i $$

$$ - b_1\beta'\,[c_2(\beta+2) + c_3(a+2)] - ab_2[c_1(\beta+2) + c_3(\beta'+2)] $$

$$-\beta b_3[c_1(\alpha+2)+c_2(\beta,+2)]$$

$$+b_1b_2d_2+b_2b_3d_3+b_1b_3d_1-b_1b_2c_3d_2(\alpha+2)(\beta'+2)$$

$$-b_2b_3c_1d_3(\alpha+2)(\beta+2)-b_1b_3c_2d_1(\beta+2)(\beta'+2)+e_1c_2c_3(\alpha)_3(\beta)_3(\beta')_2$$

$$+e_2c_1c_3(\alpha)_2(\beta)_3(\beta')_3+e_3c_1c_2(\alpha)_3(\beta)_2(\beta')_3+\prod_{i=1}^{3} e_i(\alpha_i)_4. \qquad \ldots (4.4)$$

Theorem 6: If $\alpha, \beta, \beta' > 0$, $\gamma_1, \gamma_2, \gamma_3 > 0$, $|x| < r'$, $|y| < s'$, $|z| < t'$, $r' + s' + t' + 2\sqrt{(r's't')} < 1$, then

$$1+\sum_{i=1}^{3} a_i\left[1+a_{i+1}\left(\frac{1+\alpha_{i+1}}{\alpha_{i+1}}\right)\right]+\prod_{i=1}^{3}\frac{x_i(\alpha_i)_2}{\gamma_i}$$

$$< H_B(\alpha, \beta, \beta'; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$< 1+\sum_{i=1}^{3} 2a_i\left[1+2a_{i-1}\left(\frac{1+\alpha_{i+1}}{\alpha_{i+1}}\right)\right]+8\prod_{i=1}^{3}\frac{x_i(\alpha_i)_2}{\gamma_i},$$

where

$$a_i=\frac{x_i\alpha_i\beta_i}{\gamma_i},\ b_i=\frac{(x_i)^2(\alpha_i)_2(\beta_i)_2}{2(\gamma_i)_2},\ c_i=\frac{x_i}{\gamma_i},$$

$$d_i=\frac{(\alpha_i+2)_2}{(\alpha_i)_2},\ e_i=\frac{(x_i)^2}{2(\gamma_i)_2},\ x_1=x,\ x_2=y,\ x_3=z$$

$$\alpha_1=\beta_3=\alpha,\ \alpha_2=\beta_1=\beta,\ \alpha_3=\beta_2=\beta',\quad a_4=a_1. \qquad \ldots (4.5)$$

As an example for the set of values $\alpha=.3$, $\beta=.5$, $\beta'=.7$, $\gamma_1=1.5$, $\gamma_2=2$, $\gamma_3=2.5$, $x=.1$, $y=.2$, $z=.4$, we have, from Theorem 5

$$.92639072 < H_B(\alpha, \beta, \beta'; \gamma_1, \gamma_2, \gamma_2; -x, -y, -z) < .932741$$

and from Theorem 6

$$1.08433328 < H_B(\alpha, \beta, \beta'; \gamma_1, \gamma_2, \gamma_3; x, y, z) < 1.18161824.$$

Next, approximatiag ${}_1F_1$ by Luke's results ([9], (5.5) & (5.6)) and ${}_0F_1$ by (4 1) and (4.2), we have for H_A from its integral representation ([5], p. 75 (3.1 1.1))

$$\Gamma(\alpha, \Gamma(\beta)\ H_A(\alpha, \beta, \beta'; \gamma_1, \gamma_2; x, y, z)$$

$$= \int_0^\infty \int_0^\infty e^{-(s+t)} s^{\alpha-1} t^{\beta-1}\ {}_0F_1(-; \gamma_1; xst)\ {}_1F_1(\beta'; \gamma_2; ys+zt)\ ds\ dt,$$

where $R(\alpha)$, $R(\beta)$ are positive none of γ_1 and γ_2 is a negative integer or zero and $|x| < r'$, $|y| < s'$, $|z| < t' < (1-r')(1-s')$, the inequalites

Theorem 7: If $\alpha, \beta, \beta' > 0$, $\gamma_1, \gamma_2, > 0$, $0 < \beta' < \gamma_2$, $x > 0$, $y > 0$, and $z > 0$, then

$$(1+\theta y)^{-\alpha}\ (1+\theta z)^{-\beta} \left[1 - \frac{x\alpha\beta}{\gamma_1} (1+\theta y)^{-1}\ (1+\theta z)^{-1} \right]$$

$$< H_A\ (\alpha, \beta, \beta'; \gamma_1, \gamma_2; -x, -y, -z)$$

$$< (1+\theta y)^{-\alpha}\ (1+\theta z)^{-\beta} \left[1 - \frac{x\alpha\beta}{\gamma_1} (1+\theta y)^{-1}\ (1+\theta z)^{-1} + \frac{x^2 (\alpha)_2 (\beta)_2}{2(\gamma_1)_2} \cdot \right.$$

$$\left. \cdot\ (1+\theta y)^{-2}\ (1+\theta z)^{-2} \right], \quad \ldots (4.6)$$

where $\theta = \beta'/\gamma_2$.

Theorem : If $\alpha, \beta, \beta' > 0$, $\gamma_1, \gamma_2 > 0$, $|x| < r' < 1$, $|y| < s' < 1$, and $|z| < t' < (1-r')(1-s')$, then

$$(1-\theta y)^{-\alpha} (1-\theta z)^{-\beta} \left[1 + \frac{x\alpha\beta}{\gamma_1}\ (1-\theta y)^{-1}\ (1-\theta z)^{-1} \right]$$

$$< H_A\ (\alpha, \beta\ \beta'; \gamma_1, \gamma_2; x.y, z)$$

$$< (1-\theta y)^{-\alpha} (1-\theta z)^{-\beta} \left[1 + \frac{2x\alpha\beta}{\gamma_1} (1-\theta y)^{-1}\ (1-\theta z)^{-1} \right],$$

where $\theta = \dfrac{\beta'}{\gamma_2}$. ... (4.7)

Next, application of (4.1) and (4.2) in the integral representation for H_C ([5] p. 75 (3.1.1.3))

$$\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\beta')\,H_C\,(\alpha, \beta, \beta'; \gamma; x, y, z)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(r+s+t)} r^{\alpha-1} s^{\beta-1} t^{\beta'-1} {}_0F_1(-; \gamma; xrs + yst + zrt)\, dr\, ds\, dt,$$

where $R(\alpha)$, $R(\beta)$, and $R(\beta')$ are positive, $|x| < r'$, $|y| < s'$, $|z| < t'$, $r' + s' + t' - 2\sqrt{[(1-r')(1-s')(1-t')]} < 2$,

leads to the inequalities given by

Theorem 9: If $\alpha, \beta, \beta' > 0$, $\gamma > 0$, $x > 0$, $y > 0$, and $z > 0$, then

$$1 - \frac{1}{\gamma}(\alpha\beta x + \beta\beta' y + \alpha\beta' z) < H_C\,(\alpha, \beta, \beta'; \gamma; -x, -y, -z)$$

$$< 1 - \frac{1}{\gamma}(\alpha\beta x + \beta\beta' y + \alpha\beta' z) + \frac{1}{2(\gamma)_2}\Big[x^2(\alpha)_2(\beta)_2 + y^2(\beta)_2(\beta')_2$$

$$+ z^2(\alpha)_2(\beta')_2\Big] + \frac{1}{\gamma^2}\Big[xy\alpha(\beta)_2\beta' + yz\alpha(\beta')_2\beta + xz(\alpha)_2\beta\beta'\Big], \quad \ldots (4\,8)$$

Theorem 10: If $\alpha, \beta, \beta' > 0$, $\gamma > 0$, $|x| < r'$, $|y| < s'$, $|z| < t'$, and $r' + s' + t' - 2\sqrt{[(1-r')(1-s')(1-t')]} < 2$, then

$$1 + \frac{1}{\gamma}(\alpha\beta x + \beta\beta' y + \alpha\beta' z) < H_C\,(\alpha, \beta, \beta'; \gamma; x, y, z)$$

$$1 + \frac{2}{\gamma}(\alpha\beta x + \beta\beta' y + \alpha\beta' z) \quad \ldots (4.9)$$

In particular, for the set of values $\alpha = .4$, $\beta = .7$, $\beta' = .5$, $\gamma_1 = 2.5$ $\gamma_2 = 2$, $x = .2$, $y = .3$, $z = .5$

we have from Theorem 7

$$.886317074 < H_A\,(\alpha, \beta, \beta'; \gamma_1, \gamma_2; -x, -y, -z) < .886549993,$$

and from Theorem 8

$$1.148434328 < H_A\,(\alpha, \beta, \beta'; \gamma_1, \gamma_2; x, y, z) < 1.164108269.$$ Also for the set of values $\alpha = .1$, $\beta = .8$, $\beta' = .5$, $\gamma = 2$, $x = .2$ $y = .1$ $z = .4$ we have from Theorems 9 and 10 respectively

$$.962 < H_C(\alpha, \beta, \beta'; \gamma; -x, -y, -z) < .964528$$

and

$$1.038 < H_C(\alpha, \beta, \beta'; \gamma; x, y, z) < 1.076.$$

It may also be pointed out that the integral representation ([1], p. 71)

$$G_A = G_A(\alpha, \beta, \beta'; \nu; x, y, z)$$

$$= \frac{\Gamma(\nu)}{\Gamma(\alpha)\,\Gamma(\nu-\alpha)} \int_0^1 u^{\alpha+\beta-1}(1-u)^{\nu-\alpha-1}(1-uy)^{-\beta'}(u-u^2z\cdot x)^{-\beta}\,du,$$

$$R(\nu) > R(\alpha) > 0,\ R(\alpha+\beta) > 0,\ \rho < \rho' \text{ if } |x| < \rho,\ |z| < \rho' \qquad \ldots (4.10)$$

is not valid because the condition $R(\nu-m) > R(\alpha\cdot m) > 0$, for $m \in N_0$ does not hold and therefore can not be used in obtaining the inequalities for G_A. Notabaly, however, the technique of obtaining inequalities through Lapace-type integrals which successfully yield inequalities of Lauricella-Saran functions ([13], [16]) and used in the foregoing section, does not give ineqnalities for G_A, G_B of Pandey [12] and G_C of Srivastava [15].

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FRACTIONAL DERIVATIVES OF THE LAURICELLA FUNCTIONS AND THE MULTIVARIABLE H-FUNCTION

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ABSTRACT

In the present paper, we establish four expressions for the fractional derivatives of the Lauricella functions and multivariable H-function.

1. INTRODUCTION

In this paper, the fractional derivatives of the product of the Lauricella functions and the multivariable H-function are derived. The generalized Leibniz rule has been used to establish the results concerning the product of these functions. The multivariable H-function which was introduced and studied systematically in a series of papers by Srivastava and Panda ([5], [6], [7]) is an extension of the multivariable G-function. The multivariable H-function includes Fox's H-function and Meijer's G-function of one and two variables, the generalized Lauricella function of Srivastava and Daoust [2], Appell functions, the Whittaker functions, etc. Therefore, the results established in this paper are of general character from which several known and new results can be deduced.

2. DEFINITIONS AND NOTATIONS

Following Oldham and Spanier [1, p. 49], and Srivastava and Goyal [3, p. 642, Fq. (1.1)], we define the fractional derivative of a function $f(x)$ of complex order μ (or, alternatively, a $-\mu th$ order fractional integral of $f(x)$) by

$$(2.1)\quad {}_{\alpha}D_{x}^{\mu}\,\{f(x)\} = \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_{\alpha}^{x} (x-y)^{-\mu-1} f(y)\, dy, & Re(\mu) < 0, \\ \dfrac{d^m}{dx^m}\, {}_{\alpha}D_{x}^{\mu-m}\, \{f(x)\}, & 0 \leqslant Re(\mu) < m, \end{cases}$$

where m is a positive integer

For simplicity, the special case of the fractional derivatives operator ${}_{\alpha}D_{x}^{\mu}$, when $\alpha = 0$ will be written D_{x}^{μ}.

Thus we have

$$(2.2)\quad D_{x}^{\mu} \equiv {}_{0}D_{x}^{\mu}.$$

The multivariable H-function is defined by Srivastava and Panda ([6], [7] and is represented in the manner already detailed by Srivastava, Gupta and Goyal [4, p. 251].

In this paper, we also use the short-hand notations as follows:

Following the notations given by Srivastava and Daoust [2], let

$$(2.3)\quad F^{A:\,B';\,\ldots;\,B^{(r)}}_{C:\,D';\,\ldots;\,D^{(r)}} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix} \text{ or } F\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix}$$

denote the generalized Lauricella function of several complex variables.

$$(2.3)\quad M(N,K) = \frac{(-1)^l\,(-x)^k\,(1+\alpha+\beta+2N+2K)\,(-\rho)_{n+k}\,(1+\alpha)_\rho}{m!\; l!\;(1+\alpha+\beta+N+K)_{1+\rho}\;(1+\alpha)_{n+k}}$$

$$\cdot \binom{N+\alpha+K}{N} \binom{N+\alpha+\beta+2K}{K} F_n\,[\gamma_1,\ldots\gamma_r]\,(-\xi)^{\rho-m}\,\eta^{\sigma-k-l}\,.$$

We shall also give below some results and use them later on.

(i) The well-known binomial expansion:

$$(2.5)\quad (t\ \xi)^{\lambda} = (-\xi)^{\lambda} \sum_{m=0}^{\infty} \binom{\lambda}{m} \left(-\frac{t}{\xi}\right)^{m},\ \left|-\frac{t}{\xi}\right| < 0.$$

(ii) The result [1, p. 67],

$$(2.6)\quad D_t^{\mu}(t^{\lambda}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, t^{\lambda-\mu},\ Re(\lambda) > -1.$$

(iii) The following known result of Srivastava and Panda [5, p. 9];

LEMMA. If $\mu \geqslant 0,\ 0 < t < 1,\ Re(1+\alpha) > 0,\ Re(\beta) > -1,$ $\mu_i > 0$ and $\triangle_i > 0$, or $\triangle_i = 0$ and $|z_1| < \rho_i,\ i = 1,2,\dots,r$, then

$$(2.7)\quad t^{\mu}\, F\begin{pmatrix} z_1\, t^{\mu_1} \\ \vdots \\ z_r\, t^{\mu_r} \end{pmatrix} = \sum_{N=0}^{\infty} \frac{(1+\alpha+\beta+2N)\,(-\mu)_N\,(1+\alpha)_{\mu}}{N!\,(1+\alpha+\beta+N)_{\mu+1}}$$

$$.\ F_N\,[z_1,\dots,z_r]\ {}_2F_1\left[\begin{matrix}-N,\ 1+\alpha+\beta+N; \\ 1+\alpha\ ;\end{matrix}\ t\right],$$

where

$$(2.8)\quad F_N\,[z_1,\dots,z_r] = F^{A+2:B';\dots;B^{(r)}}_{C+2:D';\dots;D^{(r)}}\left(\begin{matrix}(a):\theta',\dots,\theta^{(r)}], \\ [(c):\psi',\dots,\psi^{(r)}],\end{matrix}\right.$$

$$\begin{matrix}[1+\alpha+\mu:\mu_1,\dots\mu_r],\ [\mu+1:\mu_1,\dots,\mu_r] & :[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}]; \\ [2+\alpha+\beta+N+\mu:\mu_1,\dots\mu_r],\ [\mu-N+1:\mu_1,\dots,\mu_r] & :[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}$$

$$\left. z_1,\dots,z_r \right),\ N \geqslant 0.$$

3. THE FRACTIONAL DERIVATIVE FORMULAS

Our main results of the present paper the fractional derivative formulas involving the Lauricella functions and the multivariable H-function as follows:

$$(3.1)\quad D_t^{\mu}\,(t\ \xi)^{\rho}\, x^{\rho}\, (\eta-t)^{\rho+\sigma}\, F\begin{pmatrix}\gamma_1\ \{x\,(\eta-t)\}^{\rho_1} \\ \vdots \\ \gamma_r\ \{x\,(\eta-t)\}^{\rho_r}\end{pmatrix} H\left[\begin{matrix} z_1\{t(t-\xi)\}^{\rho_1}\{t(\eta-t)\}^{\sigma_1} \\ \vdots \\ z_r\{t(t-\xi)\}^{\rho_r}\{t(\eta-t)\}^{\sigma_r}\end{matrix}\right]$$

$$= \sum_{N,K=0}^{\infty}\ \sum_{m,l=0}^{\infty} M(N,K)\, H^{0,\ n+3:\dots;\dots;\dots}_{p+3,\ q+3:\dots;\dots;\dots}\left[\begin{matrix} z_1 t^{\rho_1+\sigma_1} \\ z_r t^{\rho_r+\sigma_r}\end{matrix} \vdots \begin{matrix}(-\xi)^{\rho_1}\ \eta^{\sigma_1} \\ (-\xi)^{\rho_r}\ \eta^{\sigma_r}\end{matrix}\right|$$

$$\left.\begin{matrix} (-\rho;\rho_1,\ldots,\rho_r), \ (-m-l;\rho_1+\sigma_1\ldots,\rho_r+\sigma_r), \ (-\sigma-k;\sigma_1,\ldots,\sigma_r),\ldots:\ldots;\ldots;\ldots \\ (m-\rho;\rho_1,\cdots,\rho_r), \ (l-\sigma\ k;\sigma_1,\ldots,\sigma_r), \ (\mu-m-l;\rho_1+\sigma_1,\ldots,\rho_r+\sigma_r),\ldots:\ldots;\ldots;\ldots \end{matrix}\right],$$

$\rho_j>0,\ \sigma_j>0,\ j=1,\ldots,r$ and $Re(\rho)+\sum_{j=1}^{r}\rho_j a_j>-1,\ Re(\sigma)+\sum_{j=1}^{r}\sigma_j a_j>-1;$

$$(3.2)\quad D_t^{\mu}(t-\xi)^{\rho}\,x^{\rho}(\eta-t)^{\rho+\sigma}\,F\begin{pmatrix} y_1\{x(\eta-t)\}^{\rho_1} \\ \vdots \\ y_r\{x(\eta-t)\}^{\rho_r} \end{pmatrix} H\begin{bmatrix} z_1\{t(t-\xi)\}^{-\epsilon}\{t(\eta-t\}^{\sigma_1} \\ z_2\{t(\eta-t)\}^{\sigma_2} \\ \vdots \\ z_r\{t\ (\eta-t)\}^{\sigma_r} \end{bmatrix}$$

$$=\sum_{N,k=0}^{\infty}\ \sum_{m,l=0}^{\infty} M(N,K)\ H^{0,\quad n+2:m_1+1,\ n_1\ ;\ldots,\ldots;\ldots}_{p+2,\ b+2:p_1+1,\ q_1+1;\ldots;\ldots;\ldots}$$

$$\left[\begin{matrix} z_1 t^{-\epsilon+\sigma_1}(-\xi)^{-\epsilon}\ \eta^{\sigma_1} \\ z_2 t^{\sigma_2}\ \eta^{\sigma_2} \\ \vdots \\ z_r t^{\sigma_r}\ \eta^{\sigma_r} \end{matrix}\ \right|$$

$$\left.\begin{matrix} (-\sigma-k;\ \sigma_1,\ldots,\sigma_r), (-m\ l;\ \sigma_1-\epsilon,\ \sigma_2,\ldots,\sigma_r),\ldots:\ldots,(1+\rho-m;\epsilon),\ldots;\ldots;\ldots \\ (l-\sigma-k;\ \sigma_1,\ldots,\sigma_r),\ (\mu-m-l;\ \sigma_1-\epsilon,\ \sigma_2\ldots\ \sigma_r),\ldots:(\rho+1;\epsilon),\ldots;\ldots;\ldots;\ldots \end{matrix}\right]$$

$\sigma_1>\epsilon>0,\ \sigma_j>0,\ j=2,\ldots,r,\ Re(\rho)>\epsilon a_1-1,\ Re(\sigma)+\sum_{j=0}^{r}\sigma_j a_j>-1;$

$$(3.3)\quad D_t^{\mu}(t-\xi)^{\rho}\,x^{\rho}(\eta-t)^{\rho+\sigma}\,F\begin{pmatrix} y_1\{x(\eta-t)\}^{\rho_1} \\ \vdots \\ y_r\{x(\eta-t)\}^{\rho_r} \end{pmatrix} H\begin{bmatrix} z_1\{t(t-\xi)\}^{\rho_1}\{t(\eta\ t)\}^{-T} \\ z_2\{t(t-\xi)\}^{\rho_2} \\ \vdots \\ z_r\{t(t-\xi)\}^{\rho_r} \end{bmatrix}$$

$$=\sum_{N,k=0}^{\infty}\ \sum_{m,l=0}^{\infty} M(N,K)\ H^{0,\ n+2:\ m_1+1,\ n_1\quad ;\ldots;\ldots}_{p+2,q+2:\ p_1+1,\ q_1+1\quad ;\ldots;\ldots}$$

$$\left[\begin{matrix} z_1 t^{\rho_1-T}\ (-\xi)^{\rho_1}\ \eta^{-T} \\ z_2 t^{\rho_2}\ (-\xi)^{\rho_2} \\ \vdots \\ z_r t^{\rho_r}\ (-\xi)^{\rho_r} \end{matrix}\ \right|$$

$$\left.\begin{matrix}(-\rho;\rho_1,\ldots,\rho_r),\ (-m-l;\rho_1-T,\rho_2,\ldots,\rho_r),\ldots:\ldots,(I-l-k+\sigma;T);\ldots;\ldots;\ldots\\(m-\rho;\rho_1,\ldots,\rho_r),\ (\mu-m-l;\rho_1-T,\rho_2;\ldots,\rho_r),\ldots:(I+k-\sigma;\tau),\ldots;\ldots;\ldots;\ldots;\ldots\end{matrix}\right]$$

$\rho_1>\tau>0,\ \rho_j>0,\ j=2,\ldots r,\ Re(\rho)+\sum_{j=1}^{r}\rho_j\alpha_j>-1,\ Re(\sigma)>T\alpha_1-1;$

$$\text{(3.4)}\quad D_t^{\mu}(t-\xi)^{\rho}x^{\rho}(\eta-t)^{\rho+\sigma}F\begin{pmatrix}y_1\{x(\eta-t)\}^{\rho_1}\\ \vdots\\ y_r\{x(\eta-t)\}^{\rho_r}\end{pmatrix}H\begin{bmatrix}z_1\{t(t-\xi)\}^{-\epsilon}\{t(\eta-t)\}^{-T}\\ z_2\\ \vdots\\ z_r\end{bmatrix}$$

$$=\sum_{N,k=0}^{\infty}\sum_{m,l=0}^{\infty}M(N,K)\,H^{0,n:\,m_1+3,\,n_1;\ldots;\ldots}_{p,q:\,p_1+3,\,q_1+3;\ldots;\ldots}\left[\begin{matrix}z_1t^{-(\epsilon+T)}(-\xi)^{-\epsilon}\eta^{-T}\\ z_2\\ \vdots\\ z_r\end{matrix}\right|\begin{matrix}\ldots:\\ \ldots:\end{matrix}$$

$$\left.\begin{matrix}\ldots,(1+\rho-m;\epsilon),(1+\sigma+k-l;T),(1+m+l-\mu;\epsilon+T);\ldots;\ldots;\ldots\\(1+\rho;\epsilon),(1+\sigma+k;T),(1+m+l;\epsilon+T),\ldots\qquad;\ldots;\ldots;\ldots\end{matrix}\right]$$

$T>0,\ \epsilon>0,\ Re(p)>\epsilon\alpha_1-1,\ Re\ \sigma)>T\alpha_1-1;$

provided $\eta\neq\xi$, $\Lambda_i>0$, $|arg\ z_i|<1/2\,\Omega_i\pi$, $\forall i\in\{1,\ldots,r\}$, where Λ_i, Ω_i and α_i as given by Srivastava, Gupta and Goyal [4, p. 252 (C. 4 & C.5), p. 253 (C.8)].

Proof. To prove (3.1), replacing the Lauricella function and the multivariable H-function by (2.7) and the Mellin-Barnes contour integral [4, p. 251 (C.1)], respectively and collecting the powers of $(t-\xi)$ and $(\eta-t)$. Finally, making use of the result (2.5) and (2.6), we get (3.1).

Similarly, (3.2), (3.3) and (3.4) can be established.

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APPLICATION OF THE MULTIVARIABLE H-FUNCTION IN HEAT CONDUCTION IN A ROD WITH ONE END AT ZERO DEGREE AND THE OTHER END INSULATED

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ABSTRACT

The use of special functions is becoming increasingly important in technical applications. Space research and nuclear reactor also give rise to several problems of the application of special functions. In this paper, an attempt has been made to discuss the application of the multivariable H-function in obtaining a solution of the partial differential equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

related to a (simple but typical) problem of heat conduction in a rod with one end at zero degree and the other end insulated. The results so derived may be found useful for computing different values of $f(x)$ for several special functions.

1. INTRODUCTION

As an example of the application of the multivariable H-function in applied mathematics, we shall consider the problem of obtaining a solution of a heat conduction problem in one dimensional rod with constant thermal properties and no sources, with one end at zero degree and the other end insulated, such that no solutions of heat equation grow exonentially in time.

Let us now formulate a simple (but typical) problem of heat flow. If the thermal coefficients are constant and there are no sources of thermal energy, then a function $u(x,t)$ representing the temperature in one-dimensional rod $0 \leqslant x \leqslant L$ must satisfy the heat equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}, \; t \geqslant 0 \qquad \ldots (1.1)$$

and the intial conditions

$$u(0,t) = 0 \qquad \ldots (1.2)$$

$$\frac{\partial u}{\partial x}(L,t, = 0 \qquad \ldots (1.3)$$

$$u(x,t) \text{ is finite when } t \to \infty \qquad \ldots (1.4)$$

$$u(x,0) = f(x) \qquad \ldots (1.5)$$

Here we shall consider

$$f(x) = (\sin\frac{\pi x}{L})^{w-1} \; H^{0,\lambda:\,(u',v')\,;\,..\,;\,(u^{(r)},v^{(r)})}_{A\,C:\,[B',D']\,;\ldots;\,[B^{(r)},D^{(r)}]} \left(\begin{matrix} z_1\,(\sin\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r\,\sin\frac{\pi x}{L})^{2k_r} \end{matrix} \right)$$

where the multivariable H-function is the H-function of several complex variables introduced by Srivastava and Panda ([4]; see also [5], p. 251).

2. THE MAIN INTEGRAL

The main integral to be evaluated here is

$$\int_0^L (\sin\frac{\pi x}{L})^{w-1} \sin(2m+1)\frac{\pi x}{L} \; H^{0,\lambda:\,(u',v')\,;\ldots;\,(u^{(r)},v^{(r)})}_{A,C:\,[B',D']\,;\ldots;\,[B^{(r)},D^{(r)}]} \left(\begin{matrix} z_1\,(\sin\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r\,(\sin\frac{\pi x}{L})^{2k_r} \end{matrix} \right) dx$$

$$= L\, 2^{1-w} \sin(2m+1)\frac{\pi}{4}\, H^{0,\lambda+1\,:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,\,C+2:[B',D'];\ldots;(B^{(r)}\,D^{(r)}]} \left(\begin{matrix} [1-w:2k_1,\ldots,2k_r], \\ [(c):\psi',\ldots,\psi^{(r)}], \end{matrix} \right.$$

$$\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(1\ 2w-2m)/4:k_1,\ldots,k_r],[(3+2m-2w)/4:k_1,\ldots,k_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}$$

$$\left. z_1^{k_1},\ldots, z_r\, 4^{k_r} \right). \qquad \ldots (2.1)$$

where $Re(w+\sum_{i=1}^{r} k_i\, d_j^{(i)}/\delta_j^{(i)}) > 0$, $k_i > 0$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant u^{(i)}$,

$|arg\,(z_i)| < T_i\pi/2$, $T_i > 0$.

The integral in (2.1) can be established easily by making use of the definition of the multivariable H-function ([4]; see also [5]) and the following result ([2], p. 372)

$$\int_0^L \left(\sin\frac{\pi x}{L}\right)^{w-1} \sin(2m+1)\frac{\pi x}{2L}\, dx = L \sin(2m+1)\frac{\pi}{4}\, \Gamma(w)\, 2^{1-w}$$

$$.\ \{\Gamma[(2w+2m+3)/4]\ \Gamma[(2w-2m+1)/4]\}^{-1}, \qquad \ldots (2.2)$$

$$Re(w) > 0.$$

3. SOLUTION OF THE PROBLEM

The solution of the problem to be established is

$$U(x,t) = 2^{2-w} \sum_{n=0}^{\infty} \sin(2n+1)\frac{\pi}{4}\, e^{-[(2n+1)\pi/2L]\mu t} \quad \sin(n+1/2)\frac{\pi x}{L}$$

$$.\ H^{0,\lambda+1\quad :(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [1-w:2k_1,\ldots,2k_r] \\ [(c):\psi',\ldots,\psi^{(r)}],[(1-2w-2n)/4:k_1,\ldots,k_r] \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(3+2n-2w)/4:k_1,\ldots,k_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \quad z_1 4^{k_1},\ldots,z_r 4^{k_r} \right) \ldots (3.1)$$

where $Re(w+\sum_{i=1}^{r} k_i\, d_j^{(i)}/\delta^{(i)}) > 0$, $|arg\,(z_i| < T_i\pi/2$, $k_i>0$, $T_i>0$,

$1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant u^{(i)}$.

The general solution of the problem stated in Section 1 on setting $u = X(x)T(t)$ can be easily obtained as ([3], p. 75)

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin(n+1/2) \frac{\pi x}{L}) \, e^{-[2n+1)\pi/2L]\mu t} \qquad \dots (3.2)$$

If $t=0$, then by virtue of (1.5), we have

$$(\sin \frac{\pi x}{L})^{w-1} H^{0,\lambda:(u',v');\dots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)})} \left(\begin{matrix} z_1 (\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r (\sin \frac{\pi x}{L})^{2k_r} \end{matrix} \right)$$

$$= \sum_{n=0}^{\infty} B_n \sin(n+1/2)\frac{\pi x}{L} \qquad \dots (3.3)$$

Multiply both sides of (3.3) by $\sin (m+1/2)\pi x/L$, integrate with respect to x from 0 to L and use (2.1) and the orthogonality property of sines; we thus obtain

$$B_m = 2^{2-w} \sin (2m+1) \frac{\pi}{4} \, H^{0,\lambda+1 \quad :(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\dots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [1-w: 2k_1,\dots,2k_r], \\ [(c): \psi',\dots,\psi^{(r)}], \end{matrix} \right.$$

$$\begin{matrix} [(a):\theta',\dots,\theta^{(r)}] & :[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}]; \\ [(1-2w-2m)/4:k_1,\dots k_r],[(3+2m-2w)/4:k_1,\dots,k_r] & :[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}$$

$$\left. z_1 4^{k_1},\dots, z_r 4^{k_r} \right) \qquad \dots (3.4)$$

Substituting the value of B_m from (3.4) in (3.2), we get the desired solution .

4. SPECIAL CASES

For $r=1$, the results in (2.1) and (3.1) reduce to the following results involving Fox's H-function ([1]):

$$\int_0^L \left(\sin \frac{\pi x}{L}\right)^{w-1} \sin (2m+1)\frac{\pi x}{L}\, H^{u,v}_{B,D}\left[z \left(\sin\frac{\pi x}{L}\right)^{2k} \middle| \begin{matrix}[(b):\phi] \\ [(d):\delta]\end{matrix} \right] dx$$

$$= L\, 2^{1-w} \sin (2m+1)\frac{\pi}{4}\; H^{u,v+1}_{B-1,D+1}\left[z4^{k} \middle| \begin{matrix}[1-w:2k], \\ [(d):\delta],[1-2w-2m/4:k],\end{matrix} \right.$$

$$\left. \begin{matrix}[(b):\phi] \\ [(3+2m-2w)/\ 4:k]\end{matrix} \right] \qquad \ldots (4.1)$$

where $k>0$, $Re(w+kd_j/\delta_j) > 0$, $1 \leqslant j \leqslant u$, $|\arg(z)| < T\pi/2$

$$\left(T = \sum_{1}^{v} \phi_i - \sum_{v+1}^{B} \phi_i + \sum_{1}^{u} \delta_i - \sum_{u+1}^{D} \delta_i\right).$$

$$u(x,t) = 2^{2-w} \sum_{n=0}^{\infty} \sin(2n+1)\,\frac{\pi}{4}\, e^{-(2n+1)\pi/2L]\mu t} \sin(n+1/2)\frac{\pi x}{L}$$

$$\cdot\, H^{u,v+1}_{B+1,D+1}\left[z\, 4^{k} \middle| \begin{matrix}[1-w:2k],\ [(b):\phi] \\ [(d):\delta],[(1-2w-2n)/4:k],[(3+2n-2w)\ /4:k]\end{matrix} \right], \quad \ldots (4.2)$$

where $Re(w+k\, d_j/\delta_j)>0$, $1 \leqslant j \leqslant u$, $|\arg(z)| < T\pi/2$, $T>0$, $k>0$.

5. CONCLUDING REMARKS

In view of the generality of the multivariable H-function involved, the results derived in this paper are of general nature and may prove to be useful in several interesting situations appearing in the literature on applied mathematics, physical sciences and engineering.

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